Deriving Rigorous Estimates of the Difference Between Euler Equation Dynamics and Semi-Geostrophic Dynamics

Mr Tom O'Neill, Department of Mathematics, University of Surrey;

Visiting: Prof Mike Cullen, Met Office;

Date of visit: 5th - 9th March 2018.

Research into how best to model atmospheric flows has advanced considerably over the last 100 years or so. It was initially proven that the atmosphere behaves much like an ideal fluid, and hence satisfies Euler's equations in three dimensions. Due to the difficulty of solving these equations, various approximations have been proposed on a range of scales; the Euler equations in two dimensions, the quasi-geostrophic (QG) equations and the semi-geostrophic (SG) equations (discussed in [3] amongst others) are three important examples of such approximations [2]. Whilst a lot is known on the existence of solutions to 2D Euler and QG (see, for example, [1], [5]) there is still a lot yet to be discovered in terms of solutions to SG, but the SG problem can also be reduced to a coupled Monge-Ampère equation coupled with a transport equation, [6], the former being fully nonlinear and hence difficult to establish good regularity and existence results, particularly at the boundary. However, some SG existence has been proven, see [4].

The main objective of this research visit was to establish the physical laws that justify approximating the Euler equations in 3 dimensions by the SG equations, when used to model the formation and movement of weather fronts. The rigorous mathematical analysis is being carried out at the University of Surrey to prove that Euler solutions (when they exist) converge to SG solutions as the Rossby number $\varepsilon \to 0$. The Met Office, and Prof Cullen in particular, have been studying the physics of the SG problem, and how we may use SG diagnostics to model weather systems that can span thousands of kms, otherwise known as a large-scale flow (see Figures below for some examples of SG diagnostics used in practice).

During my time at the Met Office, we discussed the physical application of the SG problem as well as establishing the scale of the system that was as physically relevant and mathematically tractable as possible. Working with Dr Bin Cheng at the University of Surrey, we proposed solving the problem on a periodic domain that we transform in such a way that we can use techniques from Optimal Transport Theory (see [7]) and also remain consistent with physical observations. We also attended meetings in which Prof Cullen and his colleagues discussed using the SG Diagnostics in plotting numerical forecasts of the weather, some of the results of which can be seen below.

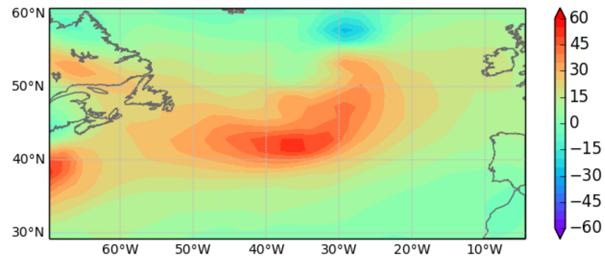

Figure 1: The geostrophic wind from a model forecast with real data over the Atlantic. This shows us the scale on which SG diagnostics are used, spanning oceans rather than analysing local weather systems.

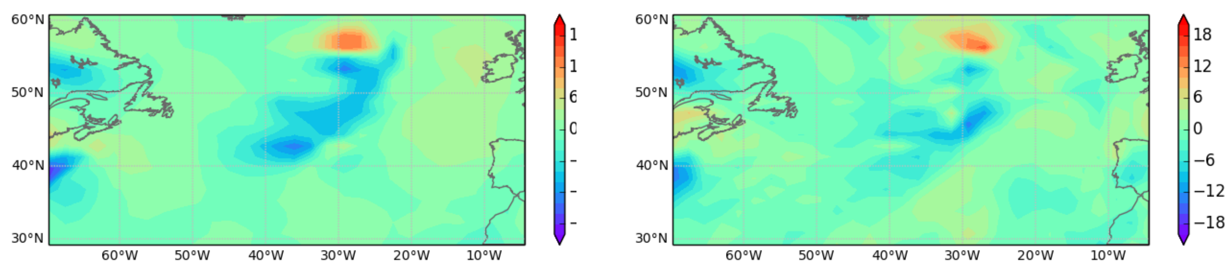

Figure 2: Left: The ageostrophic wind computed using SG diagnostics. Right: The ageostrophic wind derived from the full model. Observe that the SG diagnostic plot picks up the key peaks and general activity of the system as observed. Notice in particular the high activity at 30W, 55N detected on both plots.

References

[1] Bourgeois, A. J. and Beale, J. T. (1994). Validity of the Quasigeostrophic Model for Large-Scale Flow in the Atmosphere and Ocean. *SIAM J. Math. Anal.*, 25(4):1023–1068.

[2] Cullen, M. J. P. (2006). *A Mathematical Theory of Large-Scale Atmosphere/Ocean Flow.* Imperial College Press.

[3] Hoskins, B. J. (1975). The Geostrophic Momentum Approximation and the Semi-Geostrophic Equations. *Journal of the Atmospheric Sciences*, 32(2):233–242.

[4] Loeper, G. (2006). A Fully Nonlinear Version of the Incompressible Euler Equations: The Semi-geostrophic System. *SIAM J. Math. Anal.*, 38(3).

[5] Majda, A. J. (2003). *Introduction to PDEs and Waves for the Atmosphere and Ocean.* American Mathematical Society.

[6] Purser, R. J. and Cullen, M. J. P. (1987). A Duality Principle in Semigeostrophic Theory. *J. Atmos. Sci.*, 44:3449–3468.

[7] Villani, C. *Topics in Optimal Transprtation.* American Mathematical Society.