UKFluids SRV: multigrid solvers for high Re stationary Navier-Stokes flow

Lawrence Mitchell[1], Patrick Farrell[2], and Florian Wechsung[2]

[1] *Imperial College London*
[2] *University of Oxford*

In this short visit, we extended the Firedrake finite element library[1] to incorporate a number of features necessary for complex multigrid solvers in fluid dynamics. In addition to this, we made good progress implementing the augmented Lagrangian-based scheme of Benzi and Olshanskii [1].

The core technology improvements made to Firedrake significantly extended the existing capabilities for implementing geometric multigrid solvers, and parallel Schwarz methods. These included:

1. Grid transfer operators on all mesh types supported by Firedrake. Previously, multigrid was only available on triangular and prismatic cells. We added more general functionality providing grid transfer operators on all simplicial and tensor product cells that Firedrake supports. Of particular importance for the Navier-Stokes solver is the support for multigrid on regularly refined tetrahedral meshes.
2. Ability for the model developer to provide per-solve custom grid transfer operators. In some cases, the "natural" transfer is not sufficient for good multigrid convergence. We developed a callback system whereby application programmers can attach appropriate, problem-specific, transfer operators to individual solvers, while still taking advantage of all the other parts of Firedrake's multigrid setup.
3. Arbitrary configuration and selection of distributed mesh overlaps. For Schwarz methods, the local subdomains need more mesh halo cells than is necessary purely for global operator assembly. We extended the mesh distribution code in Firedrake to support application-guided specification of an appropriate overlap.

As well as improvements in Firedrake itself, we also extended the flexibility of the Schwarz framework developed by Mitchell[2], and used in Kirby and Mitchell [2] for primal elliptic problems. We added support for assembling patches of mixed finite element problems (for example coupled velocity-pressure systems on each patch). In addition, we also added support for a much wider variety of patch definitions. As well as existing vertex-patches, we now support patches for Vanka-like smoothers (both for MAC discretisations and continuous pressures, e.g. Taylor-Hood), morevoer, we implemented a simple callback mechanism for specifying arbitrary patches by writing a small amount of Python code. With this interface, we then added line and plane smoothers, as well as arbitrary "ordered" iterations: important for multiplicative Schwarz with strong flow directions.

This Schwarz framework is implemented as a preconditioner for PETSc[3], and we are currently working with the PETSc developers to incorporate it directly into the main PETSc library. We anticipate that it will be available to all PETSc users in the next release.

The original aim of the visit was to develop effective preconditioners for the incompressible Navier-Stokes equations for high resolution problems. Initial results are promising, we have successfully implemented the scheme of Benzi and Olshanskii [1] in two dimensions, and extended the setup to support problems in three dimensions. We have been able to solve problems of up to approximately 40 million degrees of freedom with only weak dependence

---

[1] `www.firedrakeproject.org`

[2] `github.com/wence-/ssc.git`

[3] `www.mcs.anl.gov/petsc`

of the solver on the Reynolds number. A paper is currently in preparation describing these results in full, and demonstrating the scalability on large systems (with help of an ARCHER RAP allocation). Once finalised, the solver itself will be made available as a tutorial in the Firedrake distribution.

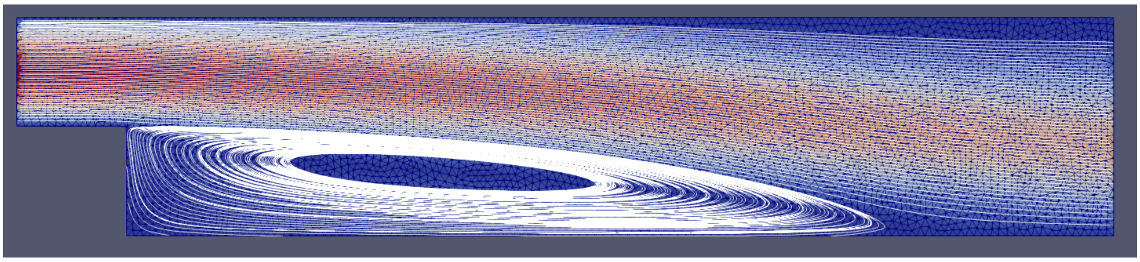

Figure 1: Two-dimensional flow over a backward facing step, solved using the preconditioner developed in this SRV.

References

[1] M. Benzi and M. A. Olshanskii. "An Augmented Lagrangian-Based Approach to the Oseen Problem". In: *SIAM Journal on Scientific Computing* 28.6 (2006), pp. 2095–2113. DOI: `10.1137/050646421`.

[2] R. C. Kirby and L. Mitchell. "Solver composition across the PDE/linear algebra barrier". In: *SIAM Journal on Scientific Computing* 40.1 (2018), pp. C76–C98. DOI: `10.1137/17M1133208`. arXiv: `1706.01346 [cs.MS]`.