Nonmodal stability analysis

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Motivation:

- bring together expertise in combustion physics, fluid dynamics, stability and control theory and identify areas of common interest

- provide training in the areas relevant to thermo-acoustic instabilities

- develop a framework for the analysis of thermo-acoustic instabilities

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A Brief History of Hydrodynamic Stability Theory

Hydrodynamic stability theory is an established and mature field of fluid dynamics concerned with the description of *disturbance behavior.*

Historical highlights

1883	Reynolds' experiment
1907/08	Orr-Sommerfeld equation
1929/33	Tollmien-Schlichting waves
1963	Compressible theory (Mack)
1965	Spatial stability theory (Gaster)
1976	Maximum energy growth (Joseph)
1983	Secondary instability theory (Orszag, Patera, Herbert)
1990	Absolute/convective instabilities (Huerre, Monkewitz)
1992	Parabolized stability equations (Bertolotti, Herbert)

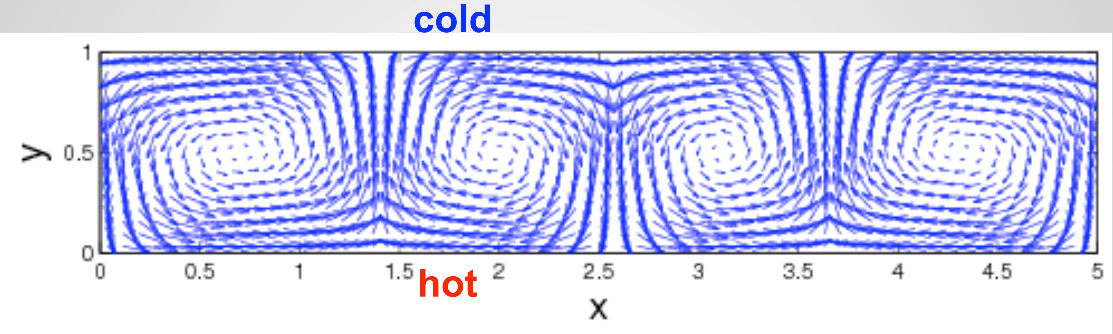
Two concepts of stability

Linear stability: we are interested in the ***minimum*** critical parameter above which a specific initial condition of ***infinitesimal*** amplitude grows ***exponentially***

Energy stability: we are interested in the ***maximum*** critical parameter below which a general initial condition of ***finite*** amplitude decays ***monotonically***

Two examples

Example 1: **Rayleigh-Bénard convection** (onset of convective instabilities can be described as an instability of the conductive state)



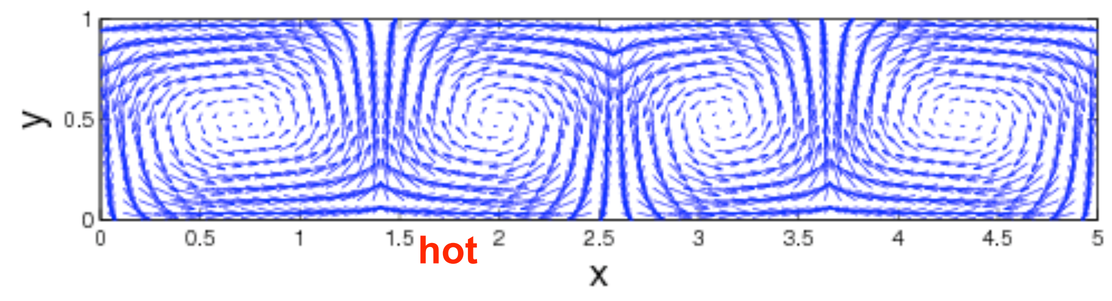
Rayleigh number (a non-dimensionalized temperature gradient) is the governing parameter

Linear stability theory: above a critical Rayleigh number of **1708** the conductive state becomes unstable to infinitesimal perturbations

Two examples

Example 1: **Rayleigh-Bénard convection** (onset of convective instabilities can be described as an instability of the conductive state)

cold



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Rayleigh number (a non-dimensionalized temperature gradient) is the governing parameter

Energy stability theory: below a critical Rayleigh number of **1708** finite-amplitude perturbations superimposed on the conductive state decay monotonically in energy

Two examples

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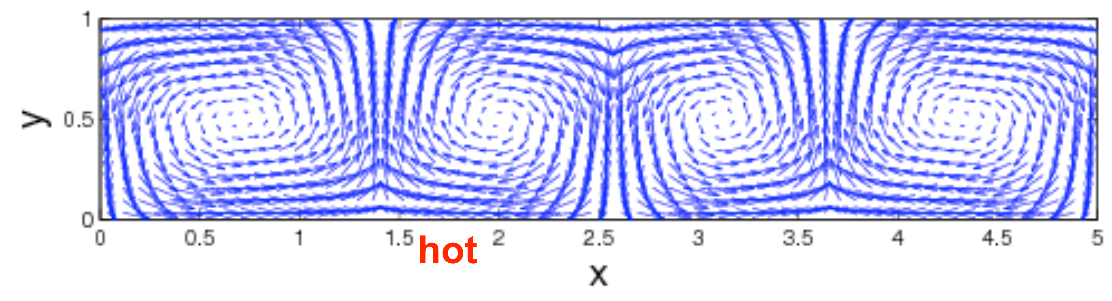

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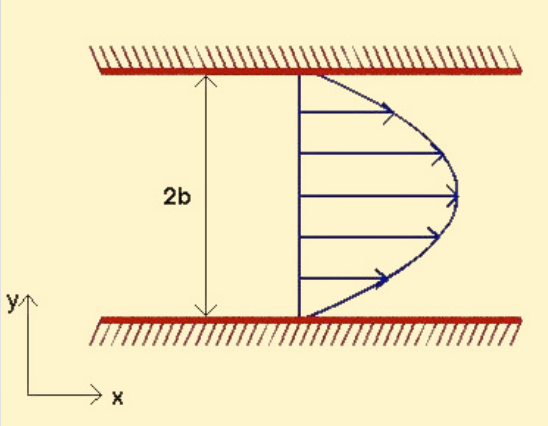
Rayleigh number (a non-dimensionalized temperature gradient) is the governing parameter

Experiments: show the onset of convective instabilities at a critical Rayleigh number of about **1710**

Two examples

Example 2: **Plane Poiseuille flow** (breakdown of the parabolic mean velocity profile)

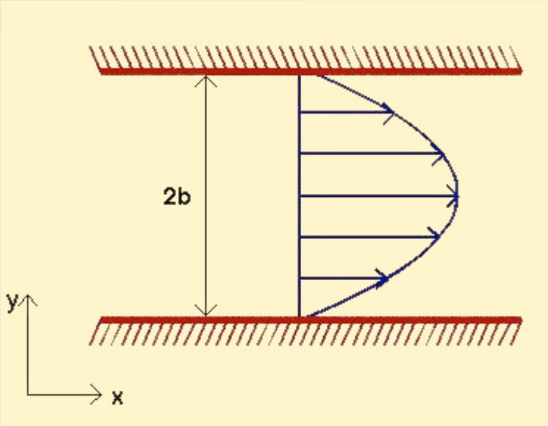


Reynolds number (a non-dimensionalized velocity) is the governing parameter

Linear stability theory: above a critical Reynolds number of **5772** the parabolic velocity profile becomes unstable to infinitesimal perturbations

Two examples

Example 2: **Plane Poiseuille flow** (breakdown of the parabolic mean velocity profile)

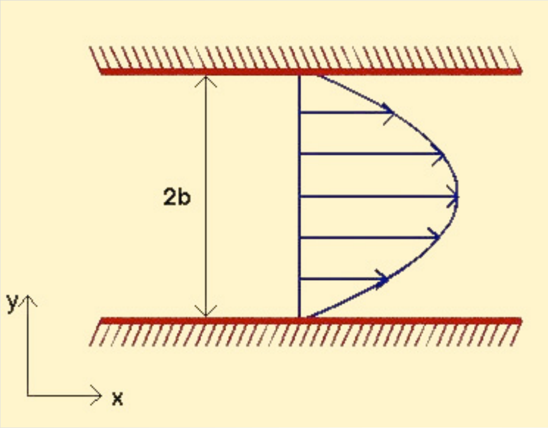


Reynolds number (a non-dimensionalized velocity) is the governing parameter

Energy stability theory: below a critical Reynolds number of **49.6** finite-amplitude perturbations superimposed on the parabolic velocity profile decay monotonically in energy

Two examples

Example 2: **Plane Poiseuille flow** (breakdown of the parabolic mean velocity profile)



Reynolds number (a non-dimensionalized velocity) is the governing parameter

Experiments: show the breakdown of the parabolic velocity profile at a critical Reynolds number of about **1000**

Two examples

Linear stability theory, energy stability theory and experiments are in excellent agreement for Rayleigh-Bénard convection

Linear stability theory, energy stability theory and experiments show significant discrepancies for plane Poiseuille flow

Questions:

Can we explain the success and failure of stability theory for the above two examples?

Is there a better way of investigating the stability of plane Poiseuille flow (and many other wall-bounded shear flows)?

A paradox

Fact:

The nonlinear terms in the Navier-Stokes equations conserve energy.

Fact:

During transition to turbulence we observe a substantial increase in kinetic perturbation energy, even for Reynolds numbers below the critical one.

Conclusion:

The increase in energy for subcritical Reynolds numbers has to be accomplished by a linear process, without relying on an exponential instability; i.e. we need *a linear instability without an unstable eigenvalue*.

Linear stability theory as a two-step procedure

Standard linear stability calculations consist of a two-step procedure:

linearization and diagonalization.

Most of the failures and shortcomings of linear stability theory have traditionally been blamed on the first step: *linearization*.

The second step, *diagonalization*, has only been questioned recently.

Linearization: the governing equations

Starting point are the Navier-Stokes equations (assuming incompressible flow)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}$$ momentum

$$\nabla \cdot \mathbf{u} = 0$$ mass

Linearization step: assuming a steady mean flow $\mathbf{U}$

decomposition of the flow field into mean and perturbation

$$\mathbf{u} = \mathbf{U} + \varepsilon \mathbf{u}'$$

Linearization: the governing equations

we obtain the linearized Navier-Stokes equations (omitting primes)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U}\nabla)\mathbf{u} + (\mathbf{u}\nabla)\mathbf{U} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

further simplifying assumptions: uni-directional mean flow dependent on one spatial coordinate, e.g.,

$$\mathbf{U} = U(y)\hat{\mathbf{x}}$$

Linearization: the governing equations

we obtain the linearized Navier-Stokes equations (omitting primes)

$$\frac{\partial \mathbf{u}}{\partial t} + U\frac{\partial \mathbf{u}}{\partial x} + U' v \hat{\mathbf{x}} = \nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

further simplifying assumptions: wave-like perturbation in the homogeneous directions

$$\mathbf{u} = \hat{\mathbf{u}}(y)\exp(i\alpha x + i\beta z)$$

Linearization: the governing equations

we obtain the linearized Navier-Stokes equations (omitting primes)

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + i\alpha U \hat{\mathbf{u}} + U' \hat{v} \hat{\mathbf{x}} = \hat{\nabla} \hat{p} + \frac{1}{Re} \hat{\nabla}^2 \hat{\mathbf{u}}$$

$$\hat{\nabla} \cdot \hat{\mathbf{u}} = \mathbf{0}$$

with

$$\hat{\nabla} = \begin{pmatrix} i\alpha \\ \mathcal{D} \\ i\beta \end{pmatrix} \qquad \hat{\nabla}^2 = \mathcal{D}^2 - \underbrace{(\alpha^2 + \beta^2)}_{k^2} \qquad \mathcal{D} = \frac{\partial}{\partial y}$$

Linearization: the governing equations

it is convenient to eliminate the pressure (and the continuity equation) by choosing the normal velocity and normal vorticity as the dependent variables

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{v}\\ \hat{\eta}\end{pmatrix} = \begin{pmatrix}\mathcal{L}_{OS} & 0\\ \mathcal{L}_{C} & \mathcal{L}_{SQ}\end{pmatrix}\begin{pmatrix}\hat{v}\\ \hat{\eta}\end{pmatrix}$$

$\mathcal{L}_{OS}$ = Orr-Sommerfeld operator

$\mathcal{L}_{SQ}$ = Squire operator

$\mathcal{L}_{C}$ = coupling operator

Linearization: the governing equations

Final step: discretization in the inhomogeneous direction (y) using spectral, compact- or finite-difference methods

$$\frac{d}{dt}\begin{pmatrix} v \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} L_{OS} & 0 \\ L_C & L_{SQ} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} v \\ \eta \end{pmatrix}}_{q}$$

$$\boxed{\frac{d}{dt}q = Lq}$$

Linearization: the governing equations

Formally, this equation has a solution in form of the matrix exponential of L.

$$\frac{d}{dt}q = Lq$$

$$q = \exp(tL)q_0$$

$$q_0 = q(t = 0)$$

The matrix exponential of L is the stability operator after the linearization step.

Linearization: the governing equations

$$\boxed{q = \exp(tL)q_0}$$

We can redefine the concept of stability based on the matrix exponential by considering the growth of perturbation energy over time.

$$G(t) = \max_{q_0} \frac{||q||^2}{||q_0||^2}$$

G(t) represents the amplification of perturbation energy maximized over all initial conditions.

Linearization: the governing equations

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We can redefine the concept of stability based on the matrix exponential by considering the growth of perturbation energy over time.

$$G(t) = \max_{q_0} \frac{||q||^2}{||q_0||^2} = \max_{q_0} \frac{|| \exp(tL)q_0||^2}{||q_0||^2}$$

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Linearization: the governing equations

$$\boxed{q = \exp(tL)q_0}$$

We can redefine the concept of stability based on the matrix exponential by considering the growth of perturbation energy over time.

$$G(t) = \max_{q_0} \frac{||q||^2}{||q_0||^2} = ||\exp(tL)||^2$$

G(t) represents the amplification of perturbation energy maximized over all initial conditions.

Diagonalization: eigenvalue analysis

In general, the matrix exponential is difficult to compute. In its place, eigenvalues of L have been used as proxies.

$$L = S \Lambda S^{-1}$$

eigenvalue decomposition

$$\| \exp(tL) \|^2 = \| \exp(tS\Lambda S^{-1}) \|^2 = \| S \boxed{\exp(t\Lambda)} S^{-1} \|^2$$

traditional stability analysis

In traditional stability analysis, the behavior of G(t) is deduced from the eigenvalues of L.

Do the eigenvalues of L capture the behavior of G(t) ?

Diagonalization: eigenvalue analysis

We can answer this question by computing upper and lower bounds (estimates) on G(t).

The energy cannot decay at a faster rate than the one given by the least stable eigenvalue λ_{max}

lower bound

$$\mathrm{e}^{2\mathrm{t}\lambda_{\mathrm{max}}} \leq \| \exp(\mathrm{tL}) \|^2$$

For the upper bound we use the eigenvalue decomposition of L.

upper bound

$$\begin{aligned} \| \exp(tL) \|^2 &= \| S \exp(t\Lambda) S^{-1} \|^2 \\ &\leq \| S \|^2 \| S^{-1} \|^2 \mathrm{e}^{2\mathrm{t}\lambda_{\mathrm{max}}} \end{aligned}$$

Diagonalization: eigenvalue analysis

We can answer this question by computing upper and lower bounds (estimates) on G(t).

$$\mathrm{e}^{2\mathrm{t}\lambda_{\max}} \leq \|\exp(\mathrm{tL})\|^2 \leq \|S\|^2\|S^{-1}\|^2\mathrm{e}^{2\mathrm{t}\lambda_{\max}}$$

Two cases can be distinguished:

$\kappa(S) = \|S\|^2\|S^{-1}\|^2 = 1$

upper and lower bound coincide: the energy amplification is governed by the least stable eigenvalue

$\kappa(S) = \|S\|^2\|S^{-1}\|^2 \gg 1$

upper and lower bound can differ significantly: the energy amplification is governed by the least stable eigenvalue ***only for large times***

Diagonalization: eigenvalue analysis

This suggests distinguishing two different classes of stability problems.

$$\kappa(S) = \|S\|^2 \|S^{-1}\|^2 = 1$$

<u>normal</u> stability problems

- orthogonal eigenvectors
- eigenvalye analysis captures the dynamics

$$\kappa(S) = \|S\|^2 \|S^{-1}\|^2 \gg 1$$

<u>nonnormal</u> stability problems

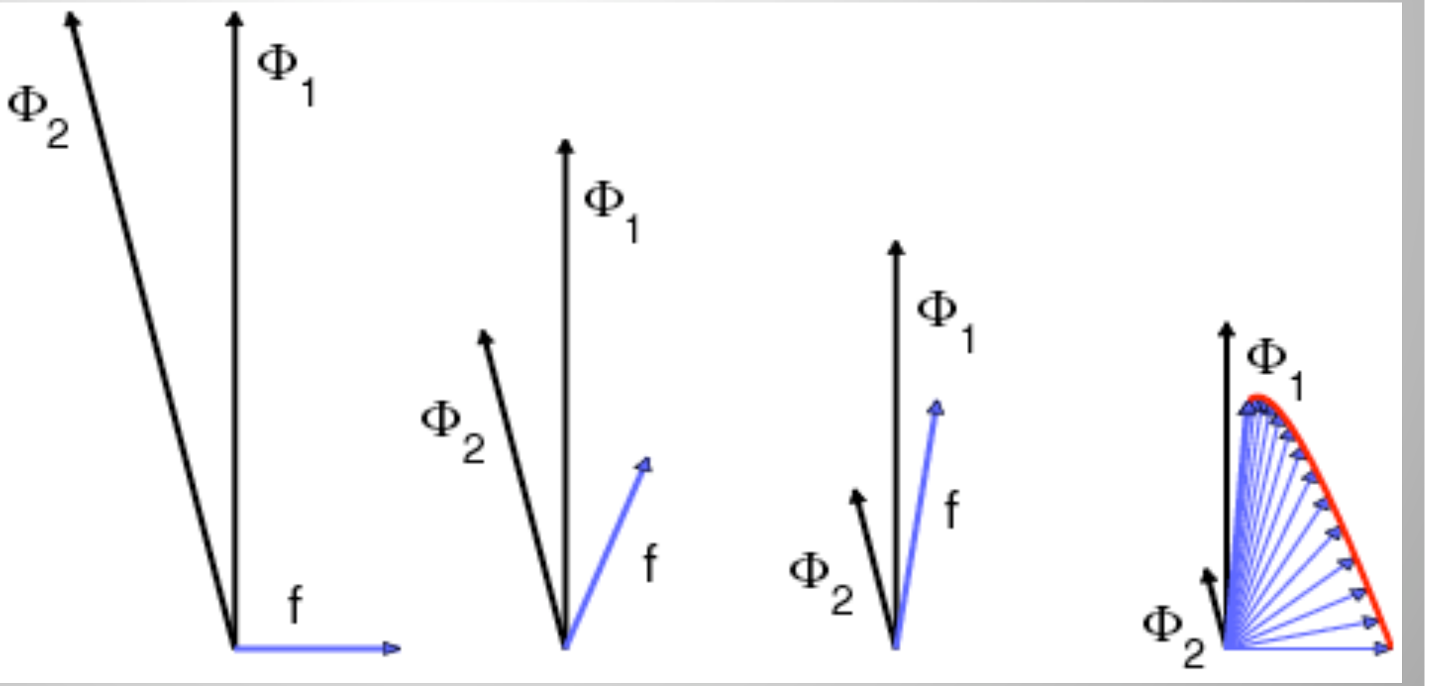
- non-orthogonal eigenvectors
- eigenvalye analysis captures the asymptotic dynamics, but not the short-time behavior

Diagonalization: eigenvalue analysis

The nonnormality of the system can give rise to transient energy amplification.

Even though we experience exponential decay for large times, the non-orthogonal superposition of eigenvectors can lead to short-time growth of energy.

Geometric interpretation:



Nonmodal stability analysis

Is there a better way of describing the short-time dynamics of nonnormal stability problems ?

$$\kappa(S) = \|S\|^2 \|S^{-1}\|^2 \gg 1$$

We start with a Taylor expansion of the matrix exponential about t=0.

$$\begin{aligned} E(t) &= \langle q, q \rangle = \|q\|^2 \\ &= \langle \exp(tL) q_0, \exp(tL) q_0 \rangle \\ &\approx \langle (I + tL) q_0, (I + tL) q_0 \rangle \\ &\approx \langle q_0, q_0 \rangle + t \ \langle q_0, (L + L^H) q_0 \rangle \end{aligned}$$

Nonmodal stability analysis

$$E(t) \approx \langle q_0, q_0 \rangle + t \, \langle q_0, (L + L^H) q_0 \rangle$$

The initial energy growth rate is given by

$$\frac{1}{E} \frac{dE}{dt}\bigg|_{t=0^+} = \frac{\langle q_0, (L + L^H) q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$(L + L^H)$ is Hermitian (symmetric)

$$\boxed{\frac{1}{E} \frac{dE}{dt}\bigg|_{t=0^+} = \lambda_{\max}(L + L^H)}$$

numerical abscissa of L

Nonmodal stability analysis

The numerical abscissa can be generalized to the ***numerical range***.

$$\frac{d}{dt}\|q\|^2 = \left\langle \frac{d}{dt}q, q \right\rangle + \left\langle q, \frac{d}{dt}q \right\rangle$$

$$= \langle Lq, q\rangle + \langle q, Lq\rangle$$

$$= 2\mathrm{Real}\,\{\langle Lq, q\rangle\}$$

Definition of the numerical range:

$$\mathcal{F}(L) = \left\{ z \mid z = \frac{\langle Lq, q\rangle}{\langle q, q\rangle} \right\}$$

set of all Rayleigh quotients of L

Nonmodal stability analysis

$$\mathcal{F}(L) = \left\{ z \mid z = \frac{\langle Lq, q\rangle}{\langle q, q\rangle} \right\}$$

set of all Rayleigh quotients of L

Three important properties of the numerical range:

1. The numerical range is convex.
2. The numerical range contains the spectrum of L.
3. For normal L, the numerical range is the convex hull of the spectrum.

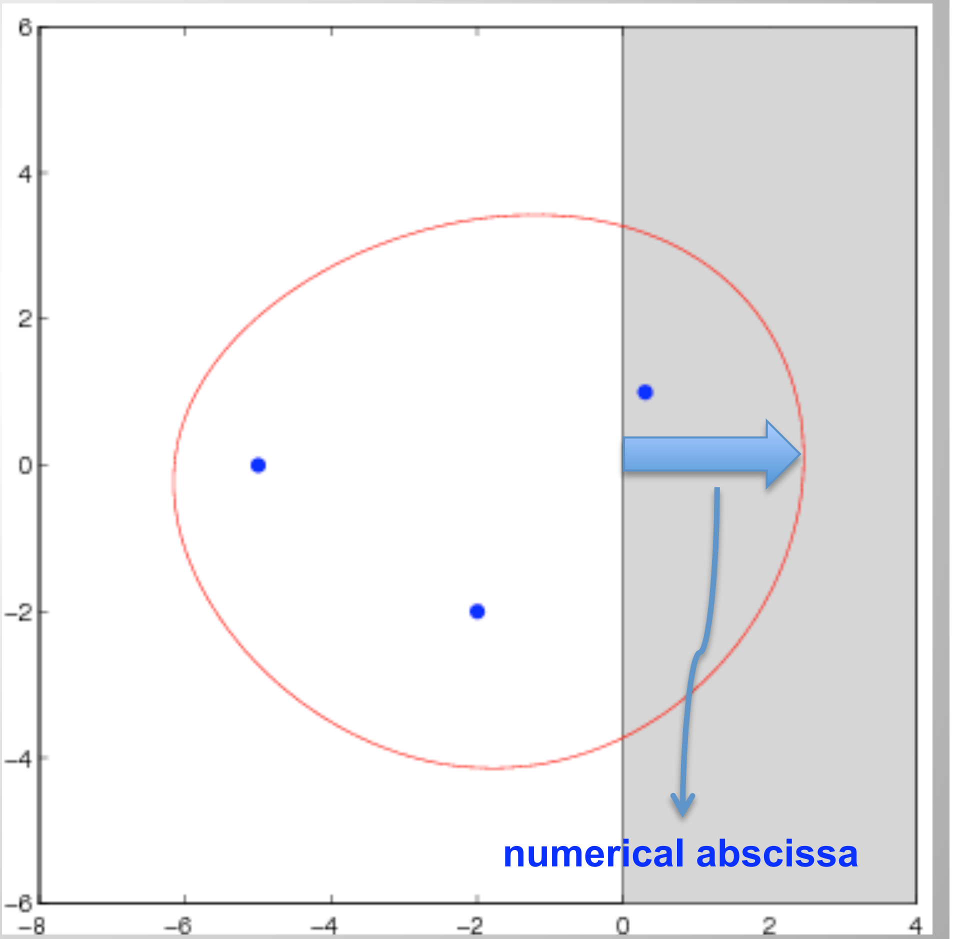
Nonmodal stability analysis

$$\mathcal{F}(L) = \left\{ z \mid z = \frac{\langle Lq, q\rangle}{\langle q, q\rangle} \right\}$$

set of all Rayleigh quotients of L

Illustration:

$$A = \begin{pmatrix} -5 & 4 & 4 \\ & -2-2i & 4 \\ & & -0.3+i \end{pmatrix}$$



The numerical range is substantially larger than the convex hull of the spectrum.

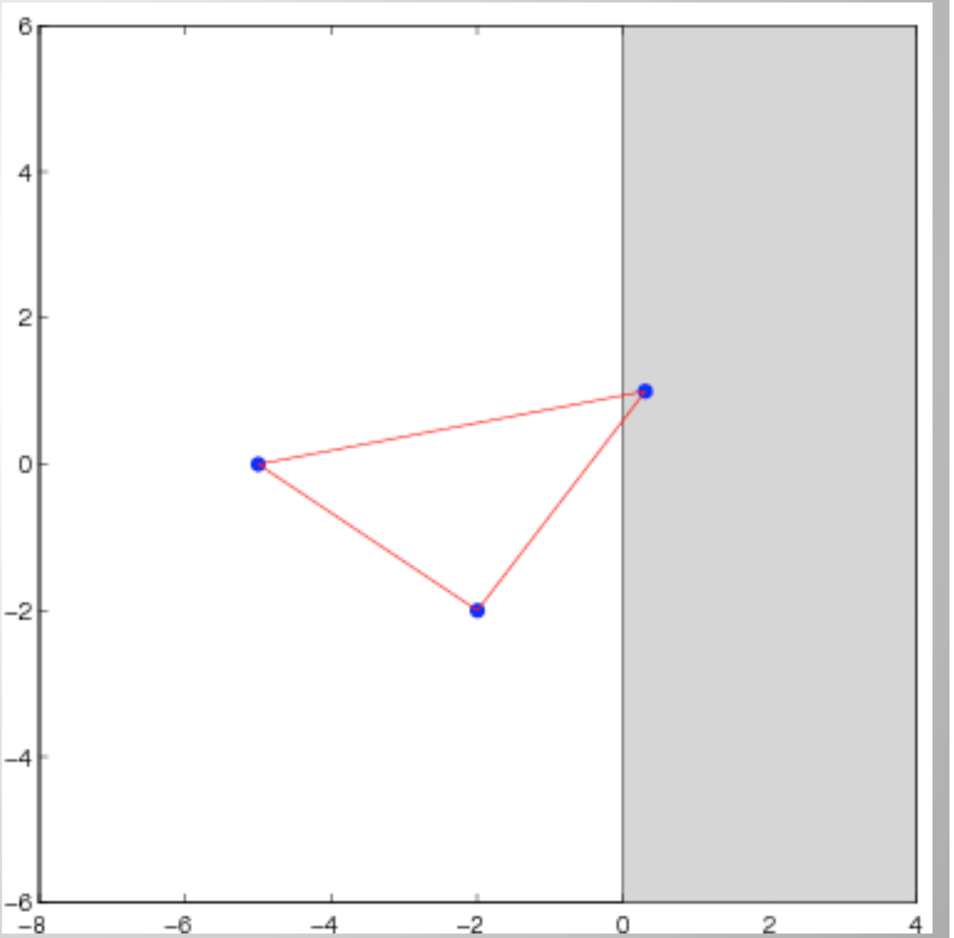
Nonmodal stability analysis

$$\mathcal{F}(L) = \left\{ z \mid z = \frac{\langle Lq, q \rangle}{\langle q, q \rangle} \right\}$$

set of all Rayleigh quotients of L

Illustration:

$$A = \begin{pmatrix} -5 & & \\ & -2-2i & \\ & & -0.3+i \end{pmatrix}$$



The numerical range is the convex hull of the spectrum.

Nonmodal stability analysis

For nonnormal stability problems:

The numerical abscissa (numerical range) governs the very short time behavior. The sign of the numerical abscissa determines initial energy growth or decay.

The least stable eigenvalue governs the long time behavior. The sign of the real part of $\lambda_{\max}$ determines asymptotic energy growth or decay.

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revisit Rayleigh-Bénard convection and plane Poiseuille flow

Nonmodal stability analysis

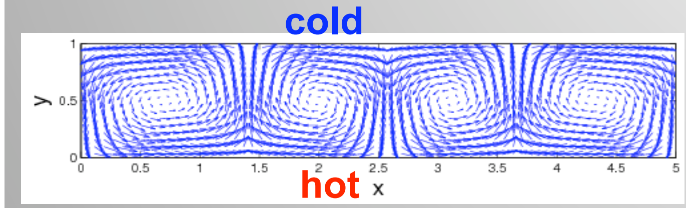
Rayleigh-Bénard convection is a **normal** stability problem

The numerical range is the convex hull of the spectrum.

The numerical range and the spectrum cross into the unstable half-plane at the same Rayleigh number.

Initial energy growth and asymptotic instability occur at the same Rayleigh number.

$$Ra_{lin} = Ra_{ener} = 1708$$

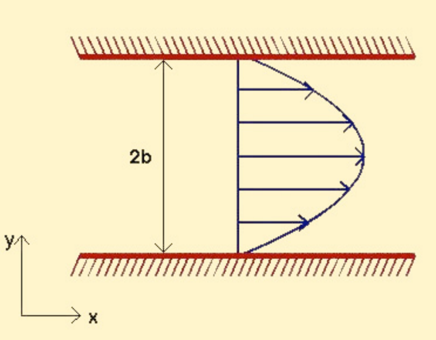


The spectrum governs the perturbation dynamics at all times.

Nonmodal stability analysis

plane Poiseuille flow is a **nonnormal** stability problem

- The numerical range is larger than the convex hull of the spectrum.
  - The numerical range crosses into the unstable half-plane « before » the spectrum crosses into the unstable half-plane.
    - Initial energy growth is possible « before » asymptotic instability occurs.
      - $$Re_{lin} = 5772 \gg Re_{ener} = 49.6$$
        - The spectrum governs the perturbation dynamics only in the asymptotic limit of $t \longrightarrow \infty$



Nonmodal stability analysis

For intermediate time, can we determine or estimate the amount of maximum transient growth?

taking the Laplace transform of the matrix exponential

$$q = \exp(tL)q_0 \Rightarrow \tilde{q} = \int_0^\infty \mathrm{e}^{-\mathrm{st}} \exp(\mathrm{tL})\mathrm{q}_0 \; \mathrm{dt} = (\mathrm{L} - \mathrm{sI})^{-1}\mathrm{q}_0$$

$$\|(L - sI)^{-1}\| \leq \int_0^\infty \|\exp(tL)\| |\mathrm{e}^{-st}| \; dt$$

$$\leq \frac{1}{\mathrm{Real}\{s\}} \max_{t \geq 0} \|\exp(tL)\|$$

$$\boxed{G_{\max} \geq \max_{\mathrm{Real}\{s\}} \mathrm{Real}\{s\} \|(L - sI)^{-1}\|}$$

lower bound for maximum transient growth (Kreiss constant)

How far does the resolvent contours protrude into the unstable half-plane?

Nonmodal stability analysis

For intermediate time, can we determine or estimate the amount of maximum transient growth?

recalling Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \oint_\Gamma f(z)(z-a)^{-1} \, dz$$

applying matrix-version of Cauchy's integral formula to exponential function

$$\exp(tL) = \frac{1}{2\pi i} \oint_\Lambda \mathrm{e}^{zt} (zI - L)^{-1} \, dz$$

$$G_{\max} \leq \frac{1}{2\pi} \oint_\Lambda \|(zI - L)^{-1}\| \, |dz|$$

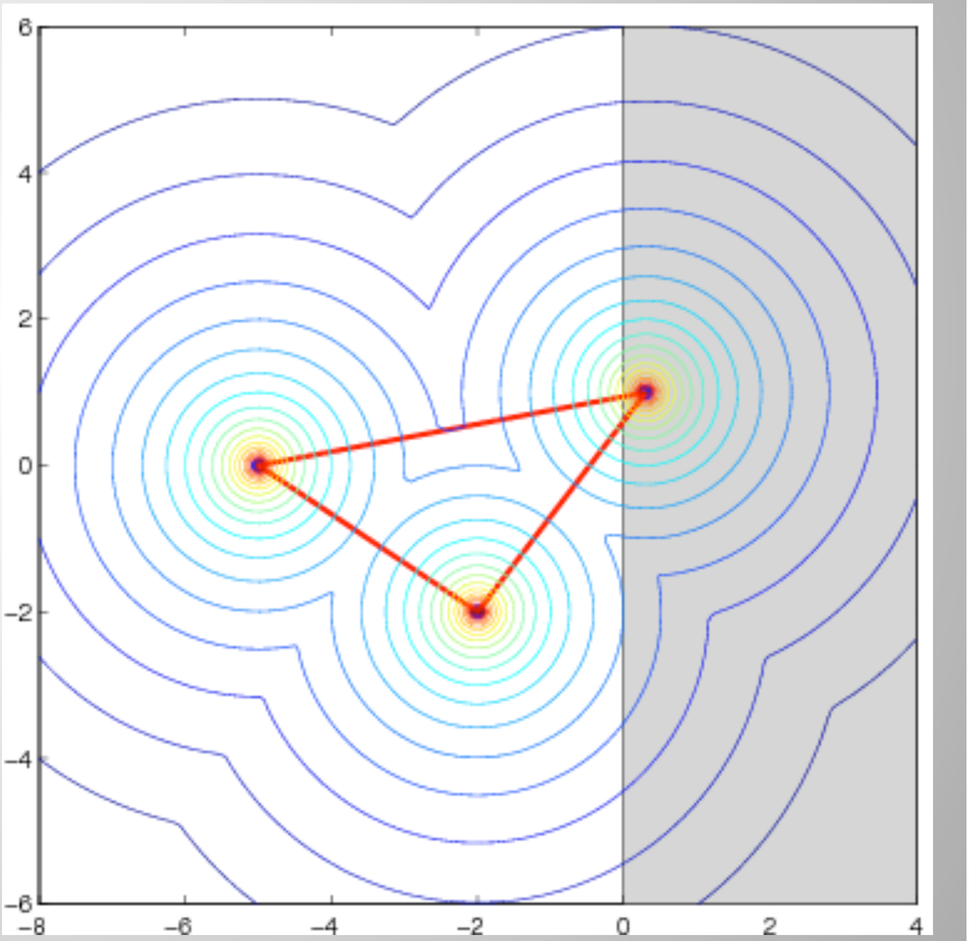
upper bound for maximum transient growth (Cauchy integral)

Nonmodal stability analysis

The resolvent norm contours for normal matrices consist of the union of disks about the eigenvalues of L.

$$A = \begin{pmatrix} -5 & & \\ & -2-2i & \\ & & -0.3+i \end{pmatrix}$$

normal matrix

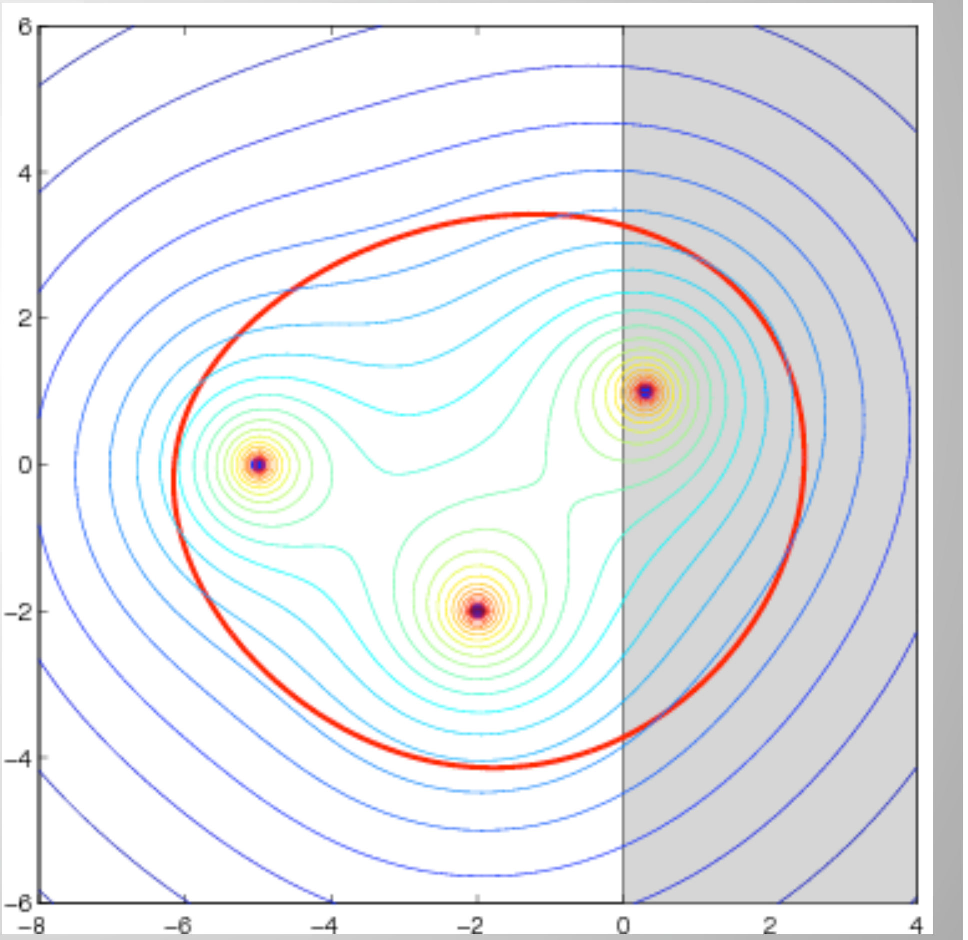


Nonmodal stability analysis

The resolvent norm contours for normal matrices consist of the union of disks about the eigenvalues of L.

$$A = \begin{pmatrix} -5 & 4 & 4 \\ & -2-2i & 4 \\ & & -0.3+i \end{pmatrix}$$

non-normal matrix



Nonmodal stability analysis

Summary

short time

$$\boxed{\frac{1}{E}\frac{dE}{dt}\bigg|_{t=0+} = \lambda_{\max}(L + L^H)}$$

(numerical abscissa)

intermediate time

$$\boxed{\max_{\mathrm{Real}\{s\}} \mathrm{Real}\{s\} \|(L - sI)^{-1}\| \leq G_{\max} \leq \frac{1}{2\pi} \oint_{\Lambda} \|(zI - L)^{-1}\| \, |dz|}$$

(resolvent norm)

long time

$$\boxed{G(t \to \infty) = \lim_{t \to \infty} \| \exp(tL) \| = \mathrm{e}^{t\lambda_{\max}}}$$

(eigenvalues)

Nonmodal stability analysis

additional interpretation of the resolvent norm: eigenvalue sensitivity

For a well-posed system we expect small perturbations to have a small effect.
Let us perturb our matrix L by random matrices of small norm and estimate the effect on the eigenvalues.

$$(L + E - \lambda I)u = 0 \qquad \|E\| = \varepsilon$$

$$(L - \lambda I)u = -Eu$$

$$\|(L - \lambda I)u\| \leq \varepsilon \|u\|$$

$$\boxed{\|(L - \lambda I)^{-1}\| \geq \varepsilon^{-1}}$$

The resolvent contours contain eigenvalues of the perturbed matrix.

Highly sensitive eigenvalues are often the « first sign » of non-normality.

Nonmodal stability analysis

The energy amplification curve G(t) is the envelope over many individual growth curves.

For each point on this curve, a specific initial condition reaches its maximum energy amplification at this point (in time).

Can we recover the initial condition that results in the maximum energy amplification at a given time? ⇨ optimal initial condition

Nonmodal stability analysis

equation that governs the optimal initial condition

$$\exp(t^* L) q_0 = q(t^*)$$

q_0 input (initial condition)

$q(t^*)$ output (final condition)

Assume that the initial condition satisfies $||q_0|| = 1$ and normalize the output such that $||\bar{q}(t^*)|| = 1$

$$\underbrace{\exp(t^* L)}_{\text{propagator}} \ \underbrace{\bar{q}_0}_{\text{input}} = \underbrace{||\exp(t^* L)||}_{\text{amplification}} \ \underbrace{\bar{q}(t^*)}_{\text{output}}$$

Nonmodal stability analysis

$$\exp(t^* L)\ \bar{q}_0 = \|\exp(t^* L)\|\ \bar{q}(t^*)$$

propagator input amplification output

The singular-value decomposition of a matrix A is

$$A = U \Sigma V^H$$

Nonmodal stability analysis

$$\underset{\text{propagator}}{\exp(t^* L)}\ \underset{\text{input}}{\bar{q}_0} = \underset{\text{amplification}}{\| \exp(t^* L)\|}\ \underset{\text{output}}{\bar{q}(t^*)}$$

The singular-value decomposition of a matrix A is

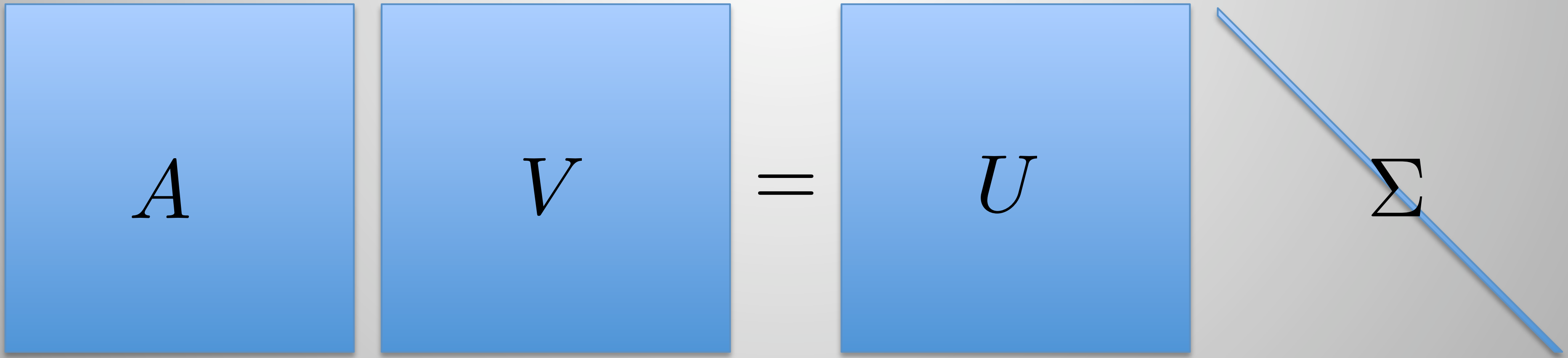
$$A = \underset{\substack{\text{unitary} \\ \text{(orthogonal)}}}{U}\ \underset{\text{diagonal}}{\Sigma}\ \underset{\substack{\text{unitary} \\ \text{(orthogonal)}}}{V^H}$$

Nonmodal stability analysis

$$\underset{\text{propagator}}{\exp(t^* L)}\ \underset{\text{input}}{\bar{q}_0} = \underset{\text{amplification}}{\| \exp(t^* L) \|}\ \underset{\text{output}}{\bar{q}(t^*)}$$

The singular-value decomposition of a matrix A is

$$AV = U\Sigma$$



Nonmodal stability analysis

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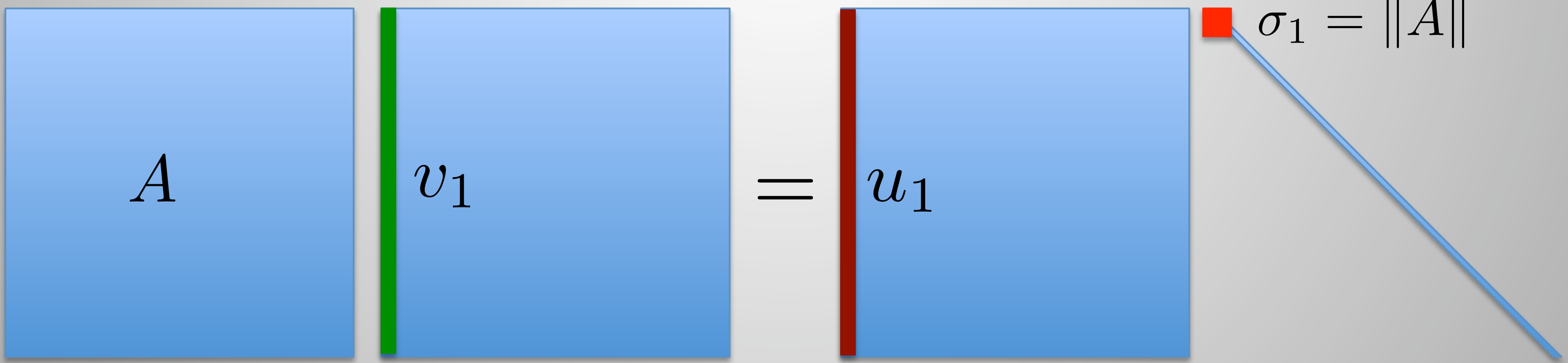
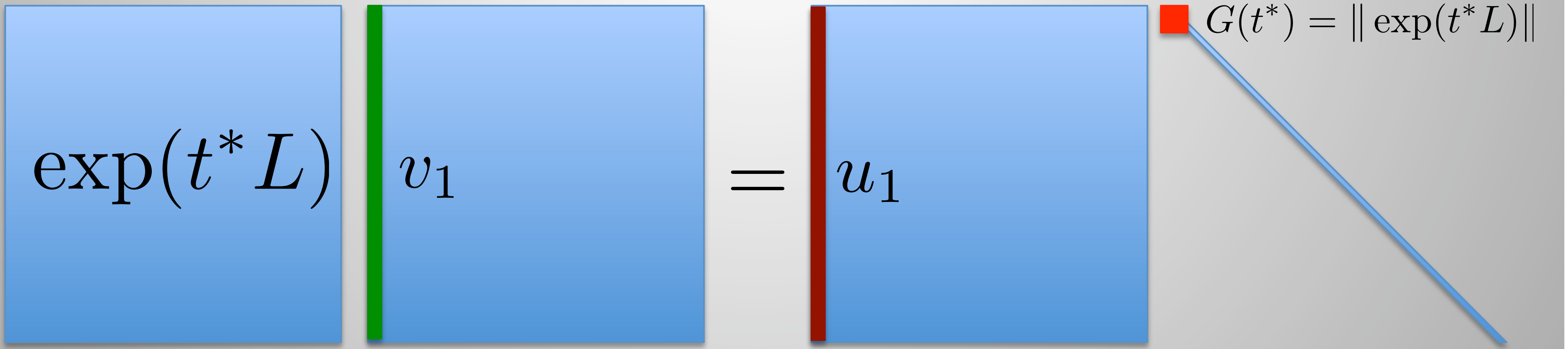


Nonmodal stability analysis

$$\underbrace{\exp(t^* L)}_{\text{propagator}} \ \underbrace{\bar{q}_0}_{\text{input}} = \underbrace{\| \exp(t^* L) \|}_{\text{amplification}} \ \underbrace{\bar{q}(t^*)}_{\text{output}}$$

The singular-value decomposition of our matrix exponential at t^* is

$$\mathrm{svd}\left(\exp(t^* L)\right) = U \Sigma V^H$$



Nonmodal stability analysis

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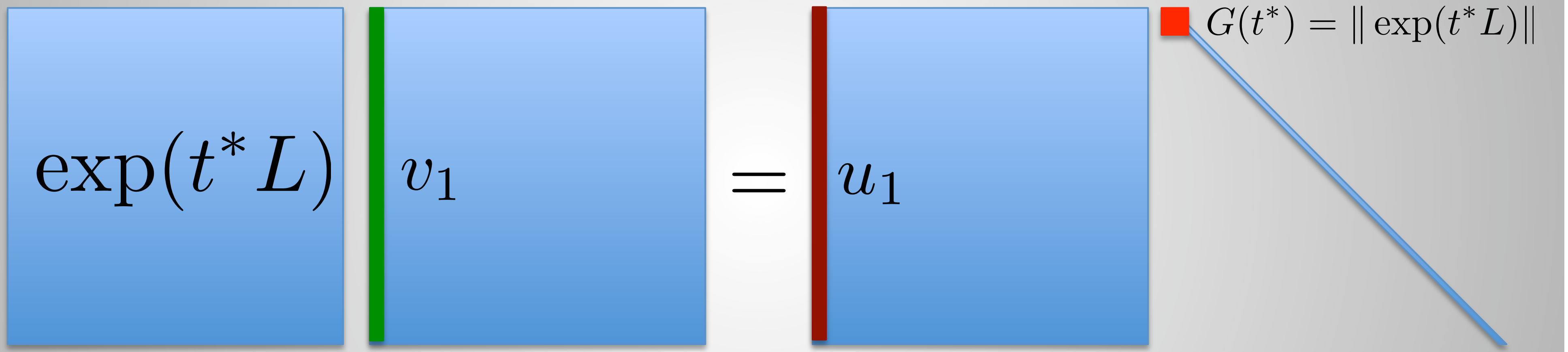


optimal initial condition
is the left principal
singular vector

Optimal final condition
is the right principal
singular vector

Nonmodal stability analysis

often we are interested in the response of our fluid system to external forces (modelling free-stream turbulence, acoustic waves, wall-roughness etc.)

in this case, our governing equation can be formulated as

$$\frac{d}{dt}q = Lq + f$$

f model of external forces

the response to forcing (particular solution, i.e., zero initial condition) is

$$q_p = \int_0^t \exp((\tau - t)L) f(\tau)\ d\tau$$

(memory integral)

Nonmodal stability analysis

for the special case of harmonic forcing $f = \hat{f}\mathrm{e}^{i\omega t}$

this simplifies to

$$\hat{q}_p = (i\omega - L)^{-1}\hat{f}$$

and the optimal response (optimized over all possible forcing functions) becomes

$$R(\omega) = \max_{\hat{f}} \frac{\|\hat{q}_p\|}{\|\hat{f}\|} = \max_{\hat{f}} \frac{\|(i\omega - L)^{-1}\hat{f}\|}{\|\hat{f}\|} = \|(i\omega - L)^{-1}\|$$

(resolvent norm)

Nonmodal stability analysis

eigenvalue-based analysis recovers the classical resonance condition

$$\|(i\omega - L)^{-1}\| = \|S(i\omega - \Lambda)^{-1}S^{-1}\| \leq \kappa(S)\frac{1}{\mathrm{dist}\{i\omega, \Lambda\}}$$

for a ***normal*** system, the classical resonance condition (closeness of forcing frequency to one of the eigenfrequencies) holds

for a ***non-normal*** system, we can have a ***pseudo-resonance*** (large response to outside forcing) even though the forcing frequency is far from an eigenfrequency of the linear system

Nonmodal stability analysis

to obtain the optimal forcing we proceed as before (i.e., take the svd)

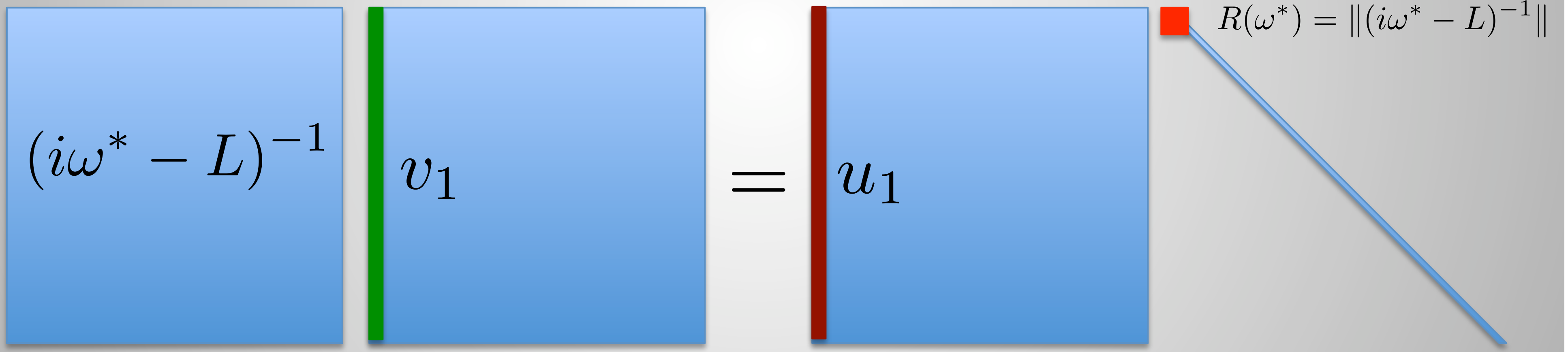
$$(i\omega^* - L)^{-1} \; \bar{f} = \|(i\omega^* - L)^{-1}\| \; \bar{q}_p$$

transfer function

forcing
(unit energy)

amplification

response
(unit energy)



$(i\omega^* - L)^{-1}$ v_1 $=$ u_1

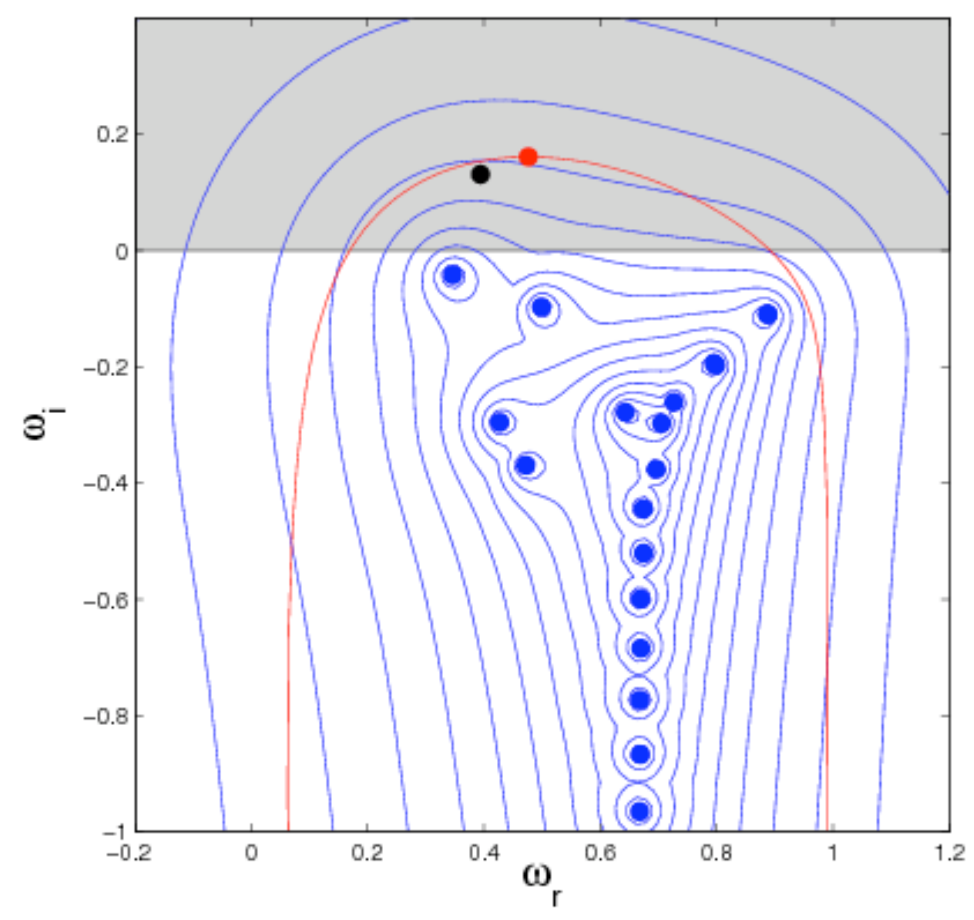
$R(\omega^*) = \|(i\omega^* - L)^{-1}\|$

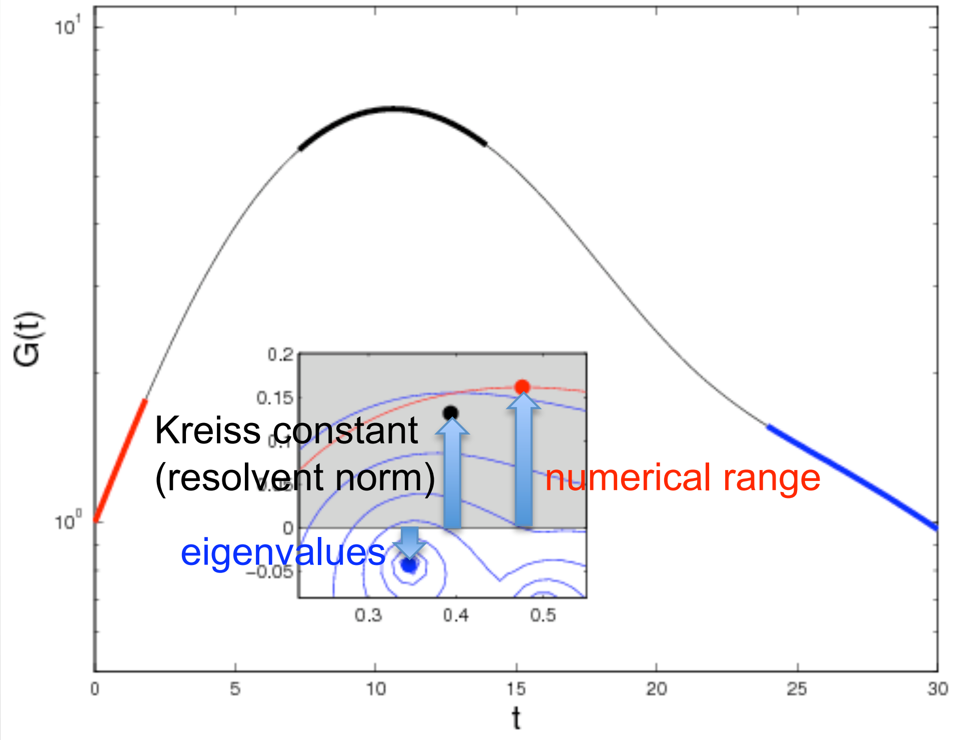
optimal harmonic forcing
is the left principal
singular vector

optimal harmonic response
is the right principal
singular vector

Nonmodal stability analysis

Example: plane Poiseuille flow





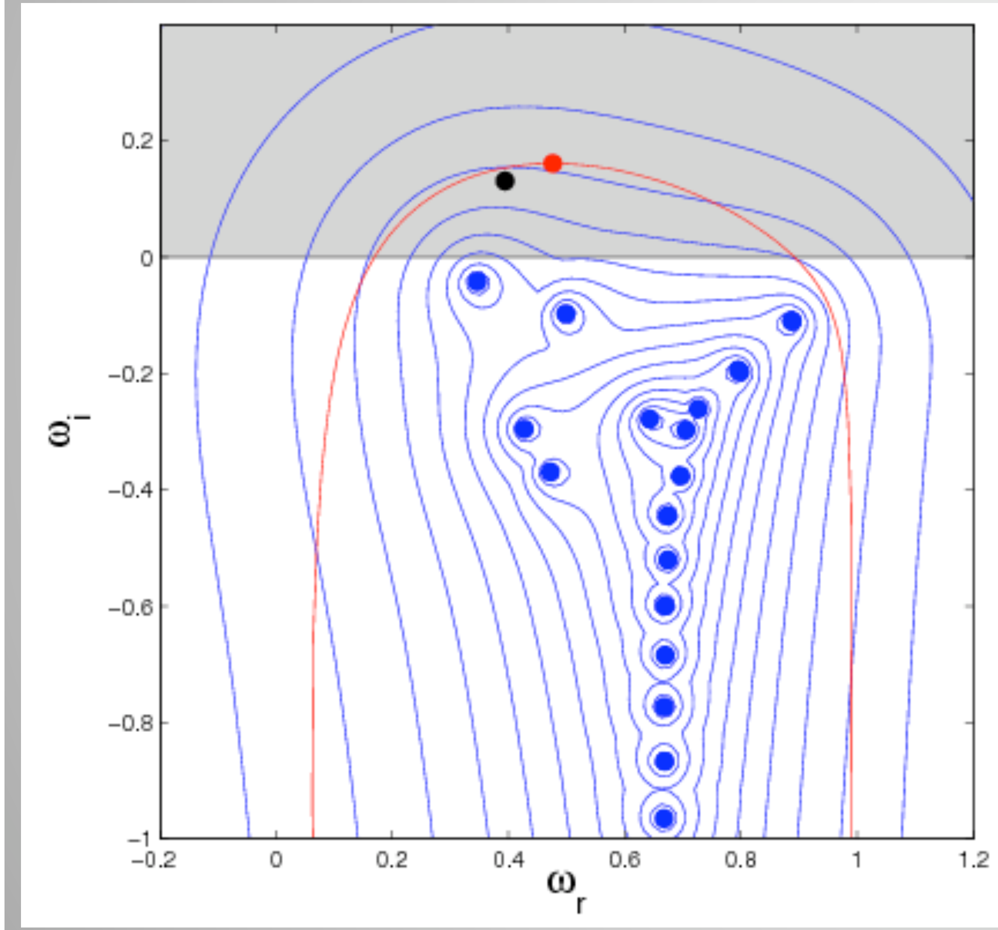
Kreiss constant
(resolvent norm)

numerical range

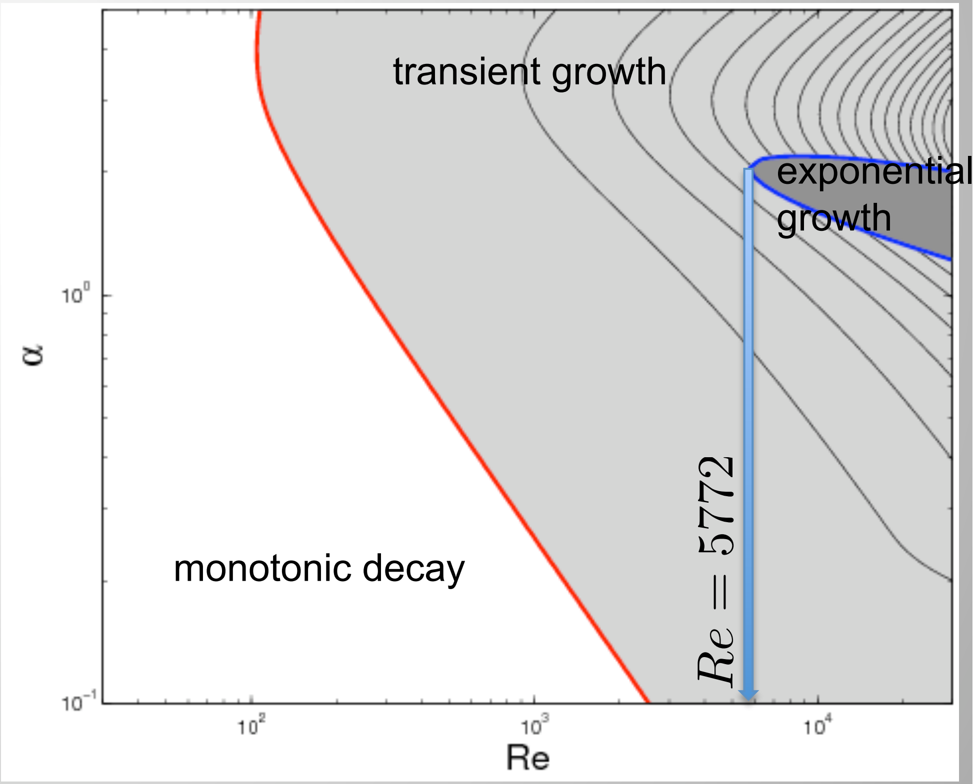
eigenvalues

Nonmodal stability analysis

Example: plane Poiseuille flow



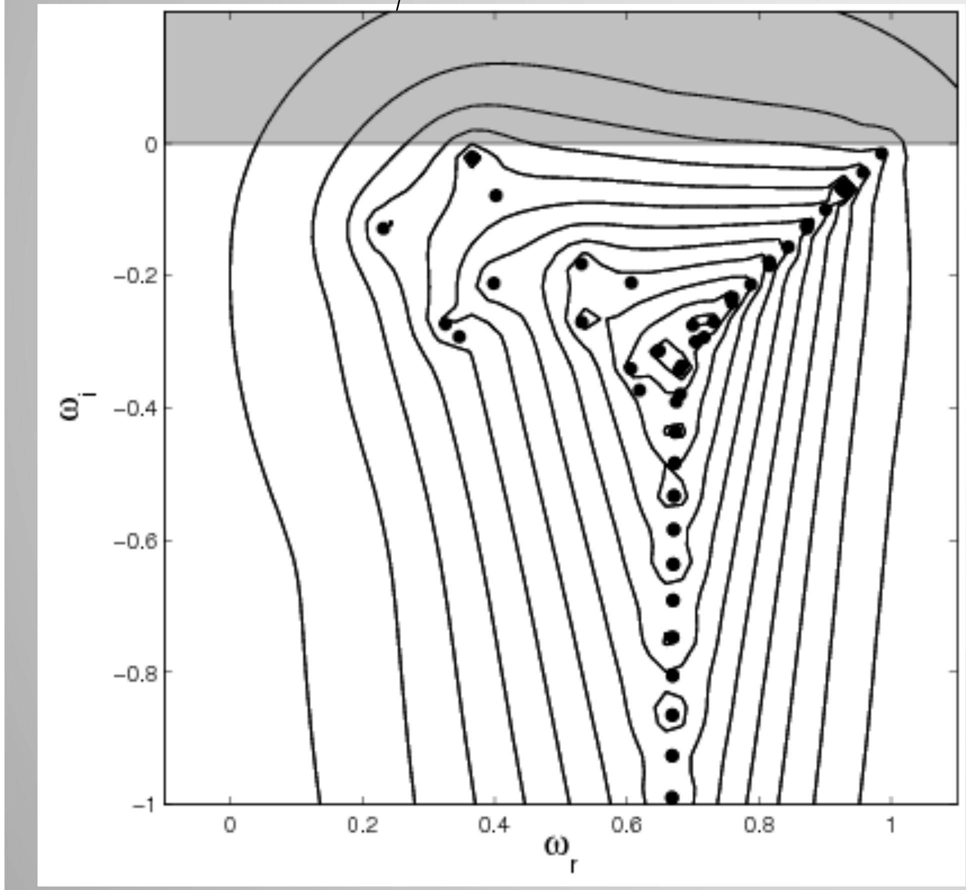
neutral curve



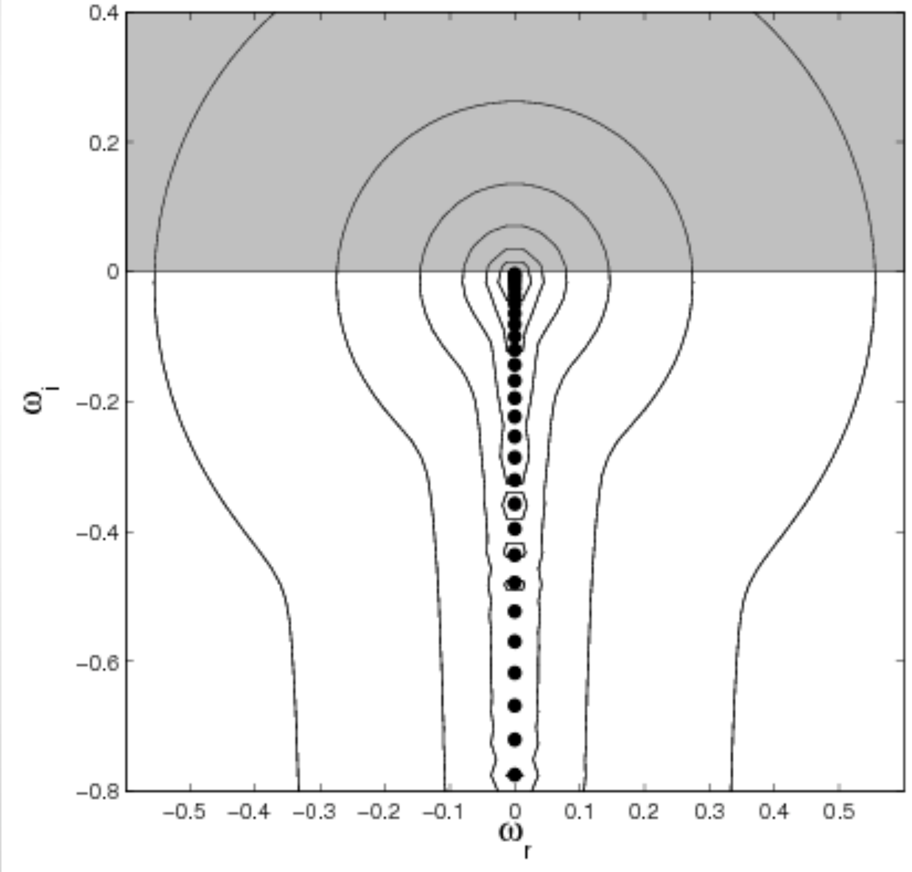
Nonmodal stability analysis

Example: plane Poiseuille flow

$\alpha = 1 \quad \beta = 1 \quad Re = 2500$



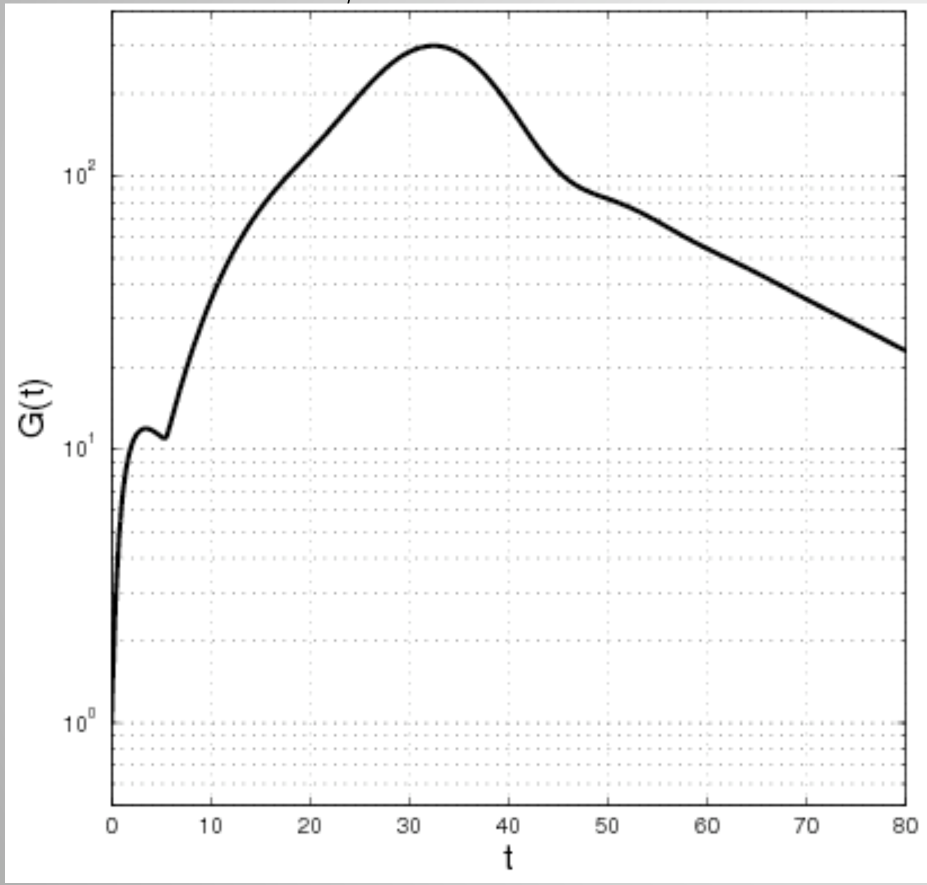
$\alpha = 0 \quad \beta = 2 \quad Re = 2500$



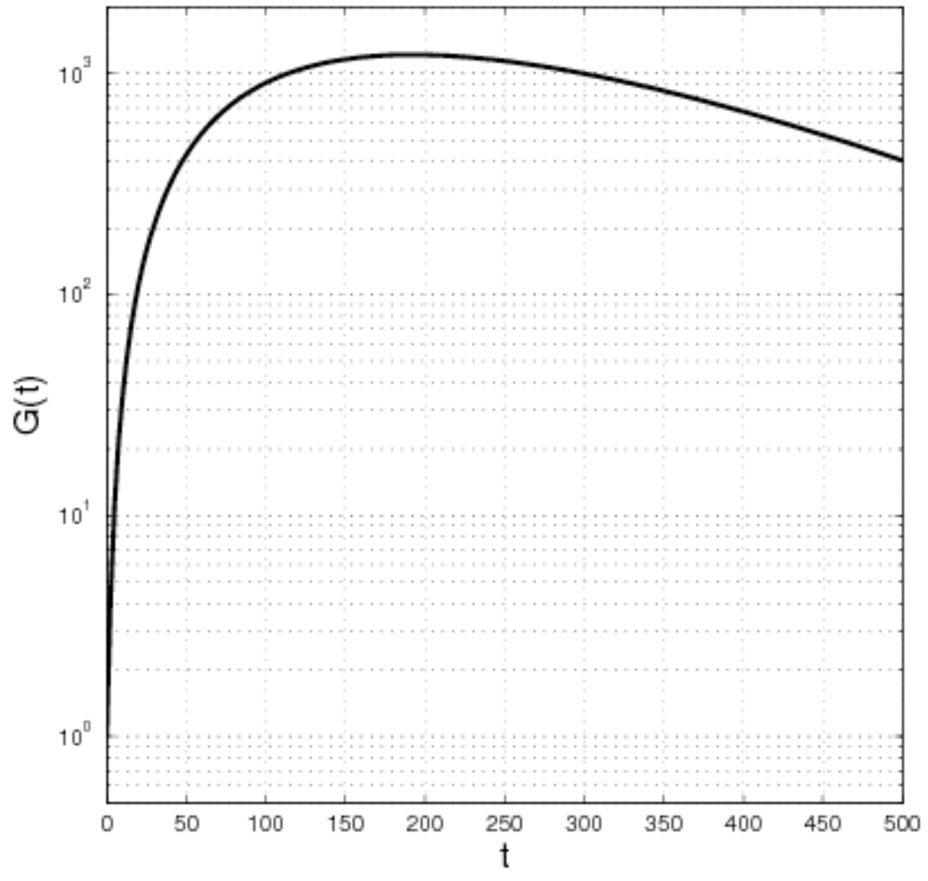
Nonmodal stability analysis

Example: plane Poiseuille flow

$\alpha = 1 \quad \beta = 1 \quad Re = 2500$

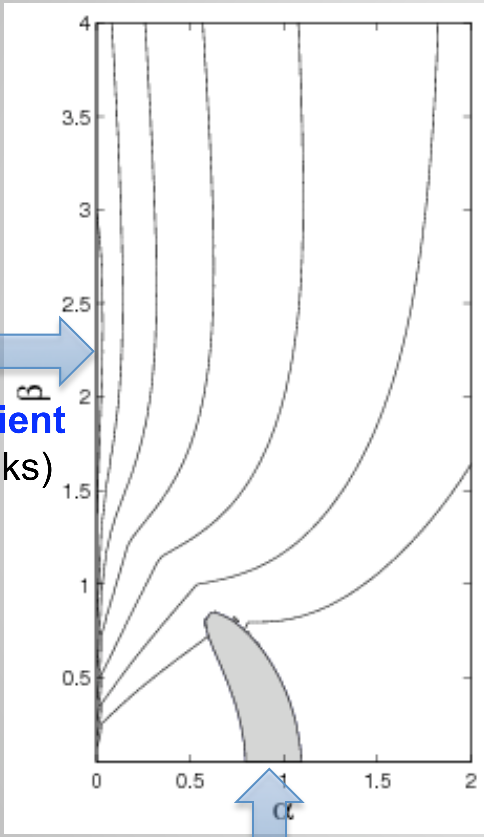


$\alpha = 0 \quad \beta = 2 \quad Re = 2500$



Nonmodal stability analysis

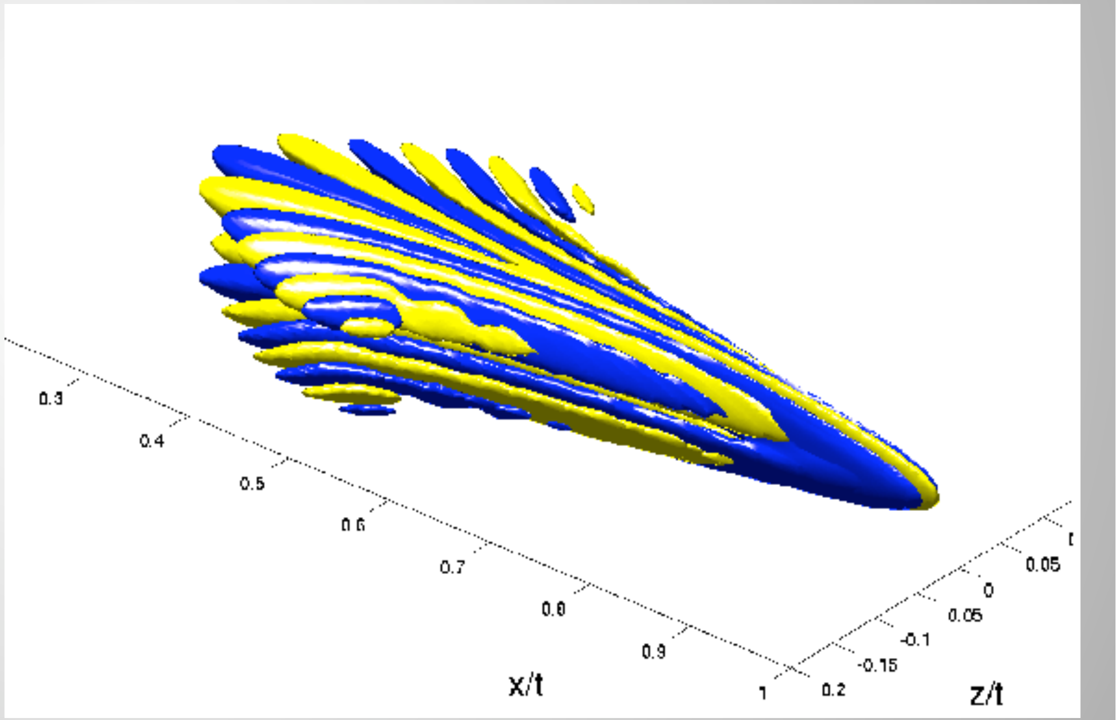
Example: plane Poiseuille flow



largest **transient** growth (streaks)

largest **exponential** growth (Tollmien-Schlichting wave)

three-dimensional impulse response



Generalizations of nonmodal stability analysis

Various generalizations of nonmodal stability analysis are possible which allow us to account for additional physical effects, such as

- stochastic forcing
- parameter and mean-flow uncertainty
- time-periodic and generally time-dependent flow
- nonlinear perturbation dynamics
- multiple inhomogeneous directions/complex geometry

Generalizations of nonmodal stability analysis

Various generalizations of nonmodal stability analysis are possible which allow us to account for additional physical effects, such as

- stochastic forcing → algebraic Lyapunov analysis
transfer functions
- parameter and mean-flow uncertainty → differential Lyapunov analysis
- time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis
- nonlinear perturbation dynamics → adjoint analysis
with check-pointing
- multiple inhomogeneous directions/complex geometry → global mode analysis

Generalizations of nonmodal stability analysis

Various generalizations of nonmodal stability analysis are possible which allow us to account for additional physical effects, such as

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  adjoint analysis
  with check-pointing
- multiple inhomogeneous directions/complex geometry → global mode analysis

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow ➡

pseudo-Floquet analysis
adjoint analysis

In many industrial applications (e.g., turbomachinery) the mean flow is periodic in time due to an oscillatory pressure gradient

We have

$$\frac{d}{dt}q = L(t)q \qquad L(t+T) = L(t)$$

period T

with the formal solution

$$\underset{\text{final solution}}{q(t)} = \underset{\text{propagator}}{A(t)}\underset{\text{initial condition}}{q_0}$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow ➡

pseudo-Floquet analysis
adjoint analysis

periodicity requires that

$$A(t+T) = A(t)\ A(T) = A(t) C$$

monodromy matrix
(mapping over one period)

$$q_n = C\ q_{n-1} = C^n\ q_0$$

initial state

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

$$q_n = C \; q_{n-1} = C^n \; q_0$$

energy amplification from period to period

$$G_n^2 = \max_{q_0} \frac{||q_n||^2}{||q_0||^2} = \max_{q_0} \frac{||C^n q_0||^2}{||q_0||^2} = ||C^n||^2$$

The eigenvalues of C are known as Floquet multipliers.

Question: Do the Floquet multipliers describe the behavior of $||C^n||^2$?

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

as before, let us compute bounds

$$\rho^{2n} \leq \|C^n\|^2 \leq \kappa^2(S)\rho^{2n}$$

largest Floquet multiplier

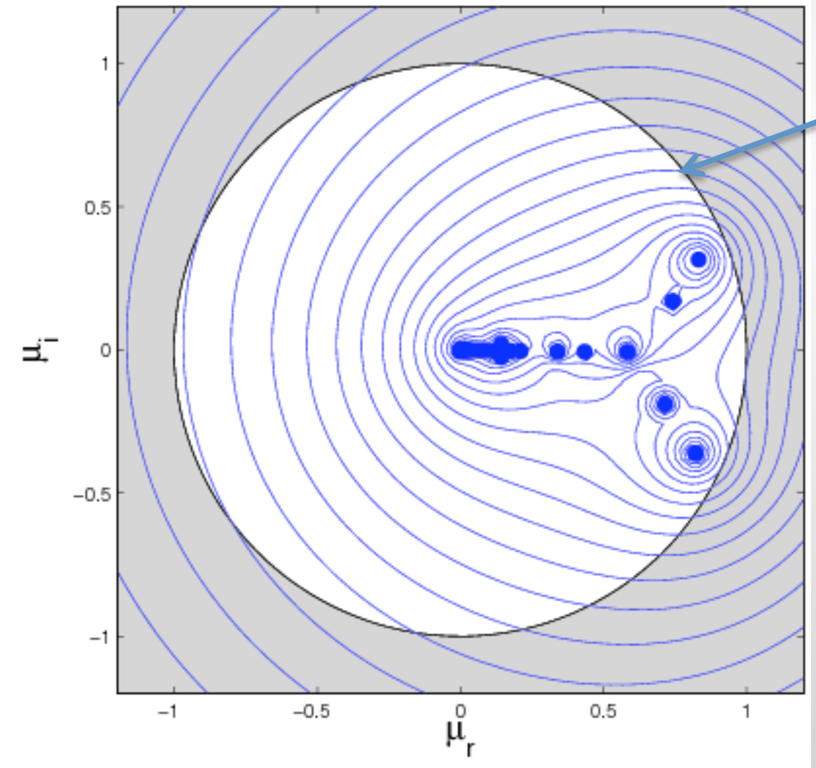
Conclusion: only for normal monodromy matrices does the largest Floquet multiplier describe the behavior from period to period

for nonnormal monodromy matrices there is a potential for transient amplification from period to period; only the asymptotic behavior $n \longrightarrow \infty$ is governed by the largest Floquet multiplier

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

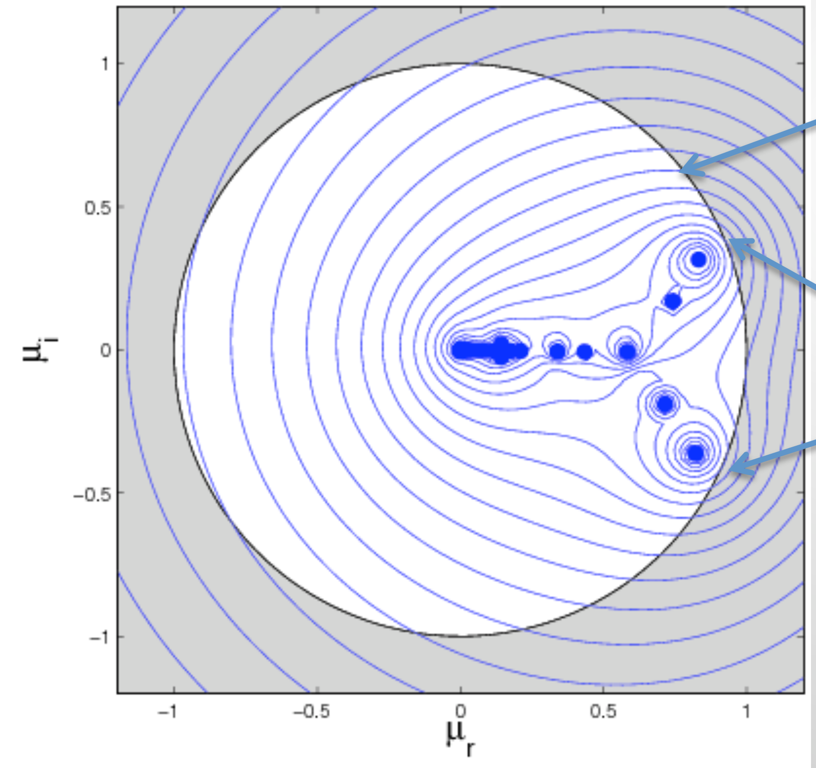


all Floquet multipliers are inside the unit disk indicating asymptotic stability (contractivity) as $n \longrightarrow \infty$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow



all Floquet multipliers are inside the unit disk indicating asymptotic stability (contractivity) as $n \longrightarrow \infty$

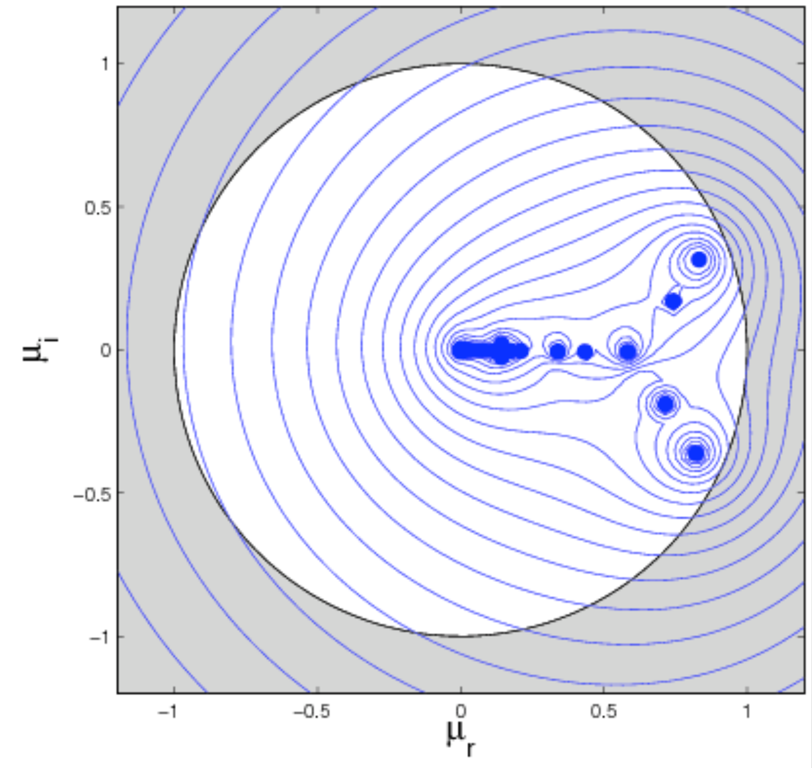
the resolvent contours reach outside the unit disk suggesting initial transient growth from period to period

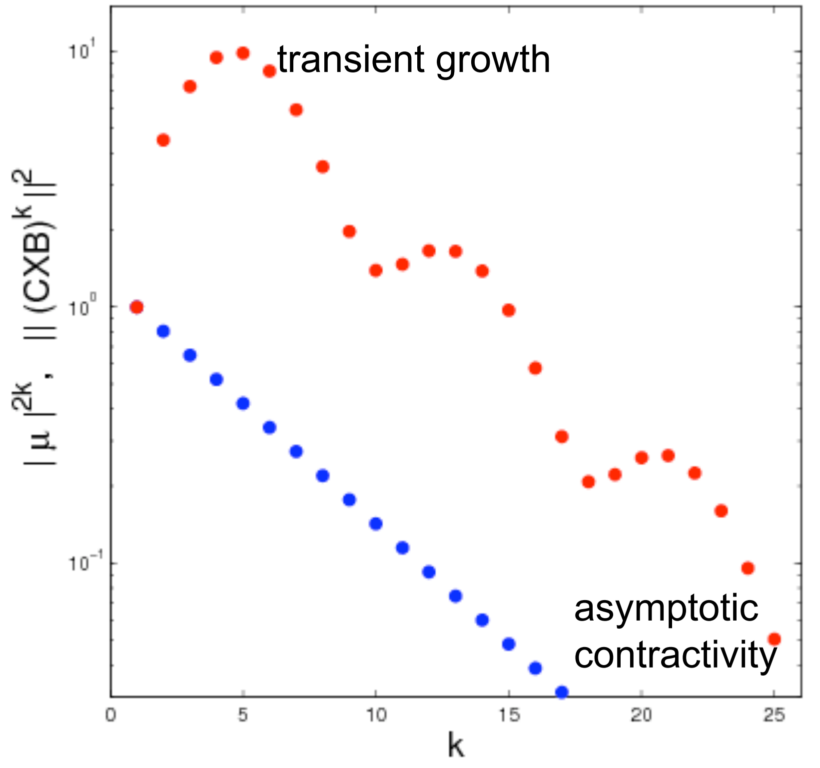
Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow




transient growth

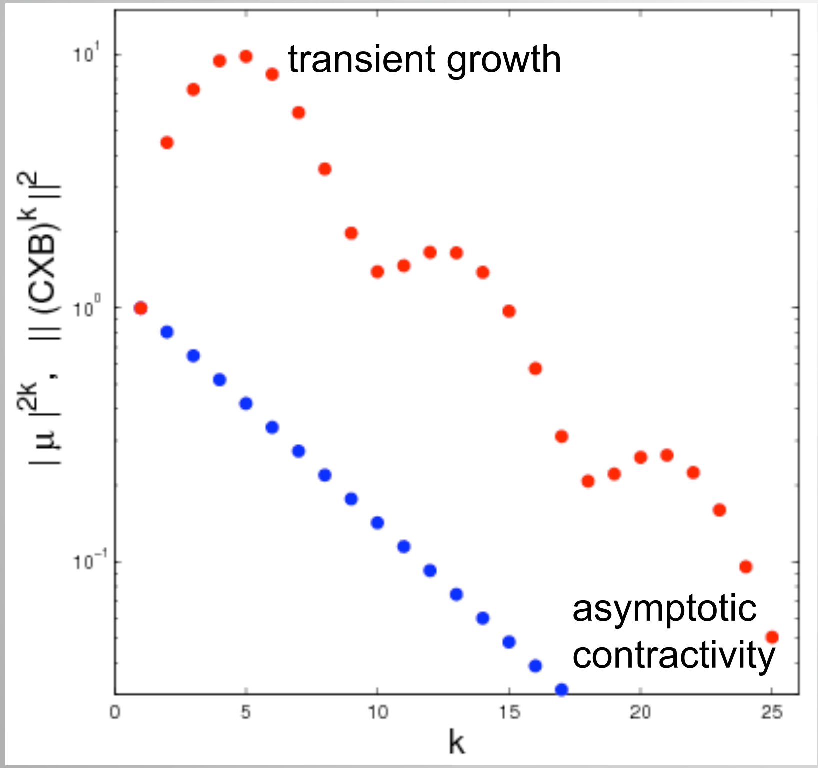
asymptotic contractivity

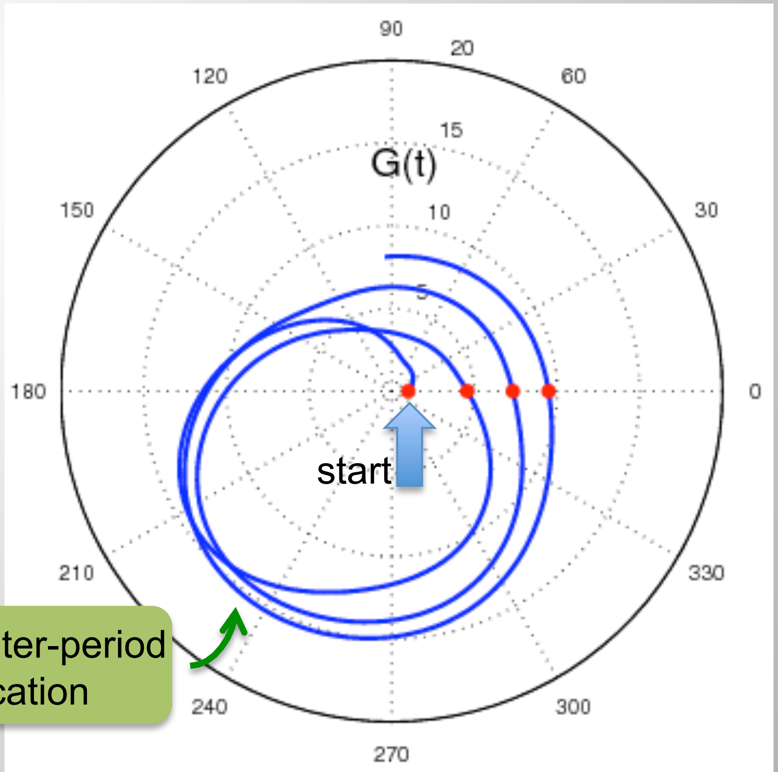
Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow





large inter-period amplification

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

Can we analyze the amplification of energy between one period, i.e., for a non-periodic system matrix ?

We have

$$\frac{d}{dt} q = L(t) q$$

with the formal solution

$$q(t) = A(t) \; q_0 \ \text{initial condition}$$

final solution propagator

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

We can formulate the optimal amplification of energy as

$$G(t)^2 = \max_{q_0} \frac{\langle q, q \rangle}{\langle q_0, q_0 \rangle}$$

$$= \max_{q_0} \frac{\langle A(t) q_0, A(t) q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$$= \max_{q_0} \frac{\langle A^H(t) A(t) q_0, q_0 \rangle}{\langle q_0, q_0 \rangle}$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow ➡

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

$$G(t)^2 = \max_{q_0} \frac{\langle A^H(t) A(t) q_0, q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$A^H A$ is a ***normal*** matrix

- the maximum is achieved for the principal eigenvector of $A^H A$
- the principal eigenvector (and eigenvalue) can be found by power iteration

$$q_0^{(n+1)} = \rho^{(n)} A^H A \; q_0^{(n)}$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

$$q_0^{(n+1)} = \rho^{(n)} A^H A \, q_0^{(n)}$$

break to power iteration into two pieces

first step

$$w(t) = A \, q_0^{(n)}$$

propagation of initial condition forward in time

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

$$q_0^{(n+1)} = \rho^{(n)} A^H A \; q_0^{(n)}$$

break to power iteration into two pieces

second step $$q_0^{(n+1)} = \rho^{(n)} A^H(t) w(t)$$

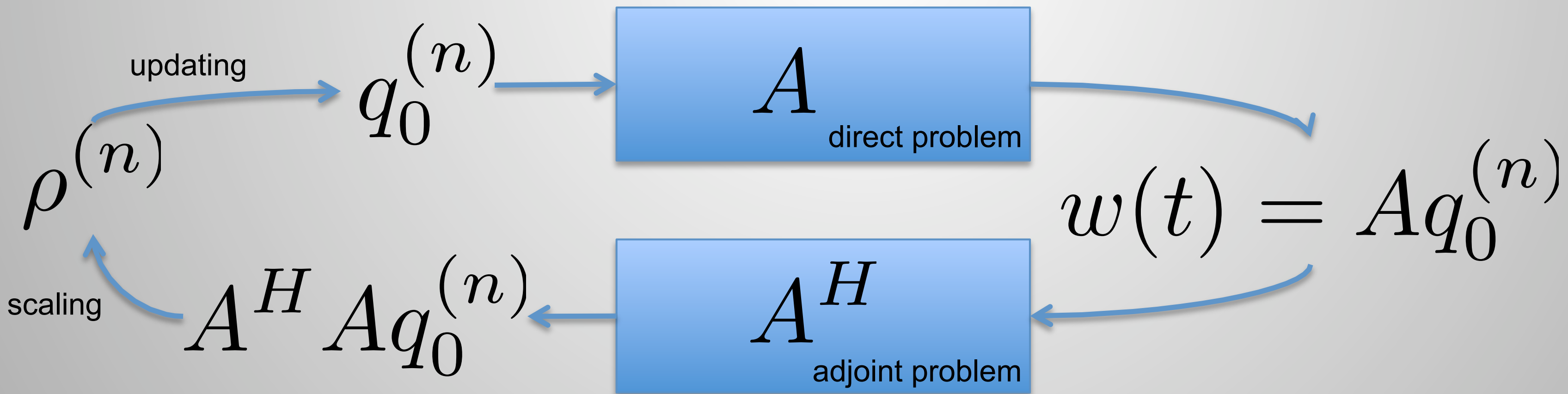
propagation of final condition backward in time

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

$$q_0^{(n+1)} = \rho^{(n)} A^H A \, q_0^{(n)}$$



updating

$q_0^{(n)}$

A

direct problem

$w(t) = A q_0^{(n)}$

A^H

adjoint problem

$A^H A q_0^{(n)}$

scaling

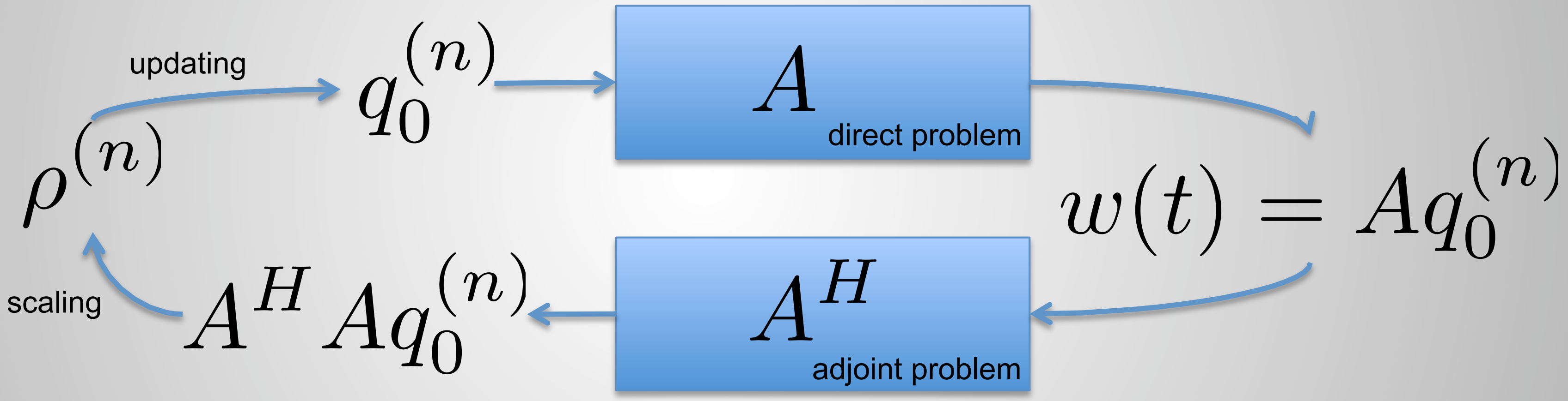
$\rho^{(n)}$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow ➡

pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow



updating

$q_0^{(n)}$

A

direct problem

$w(t) = Aq_0^{(n)}$

A^H

adjoint problem

$A^H A q_0^{(n)}$

scaling

$\rho^{(n)}$

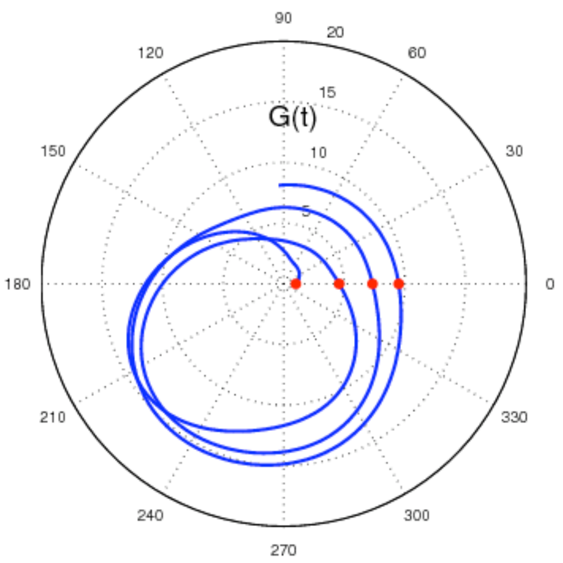
A can be any discretized solution operator. The above technique (adjoint looping) can be applied to general time-dependent stability problems.

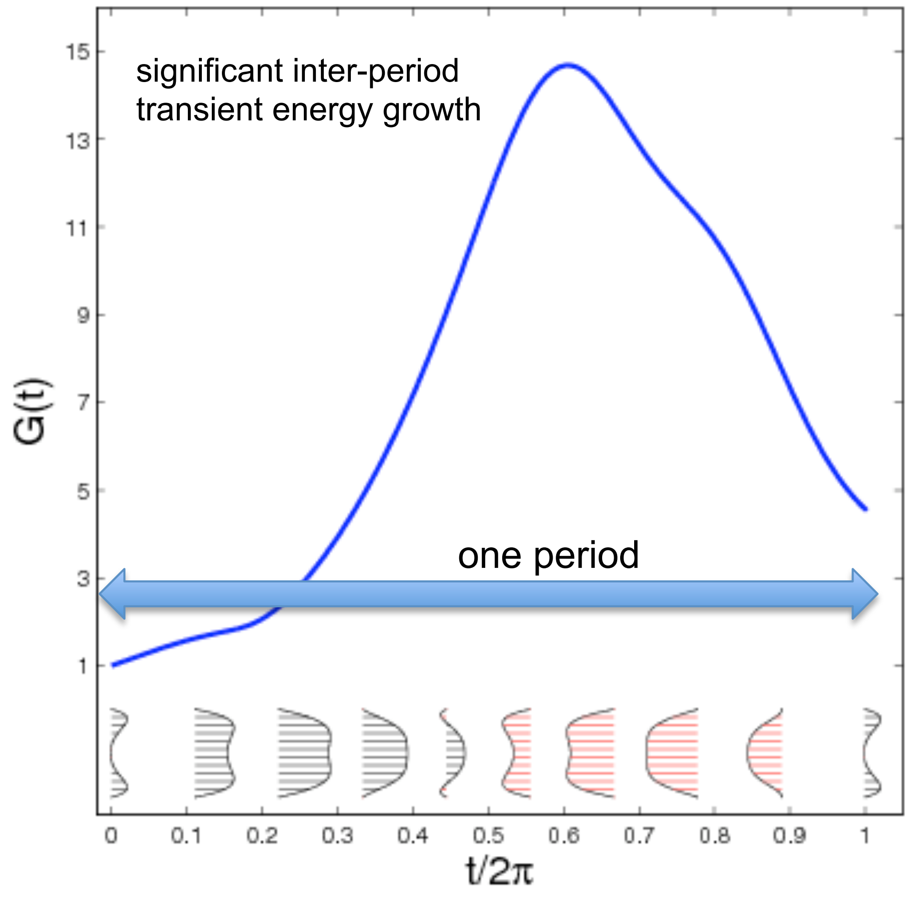
Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

applying adjoint looping to the pulsatile (inter-period) stability problem




significant inter-period transient energy growth

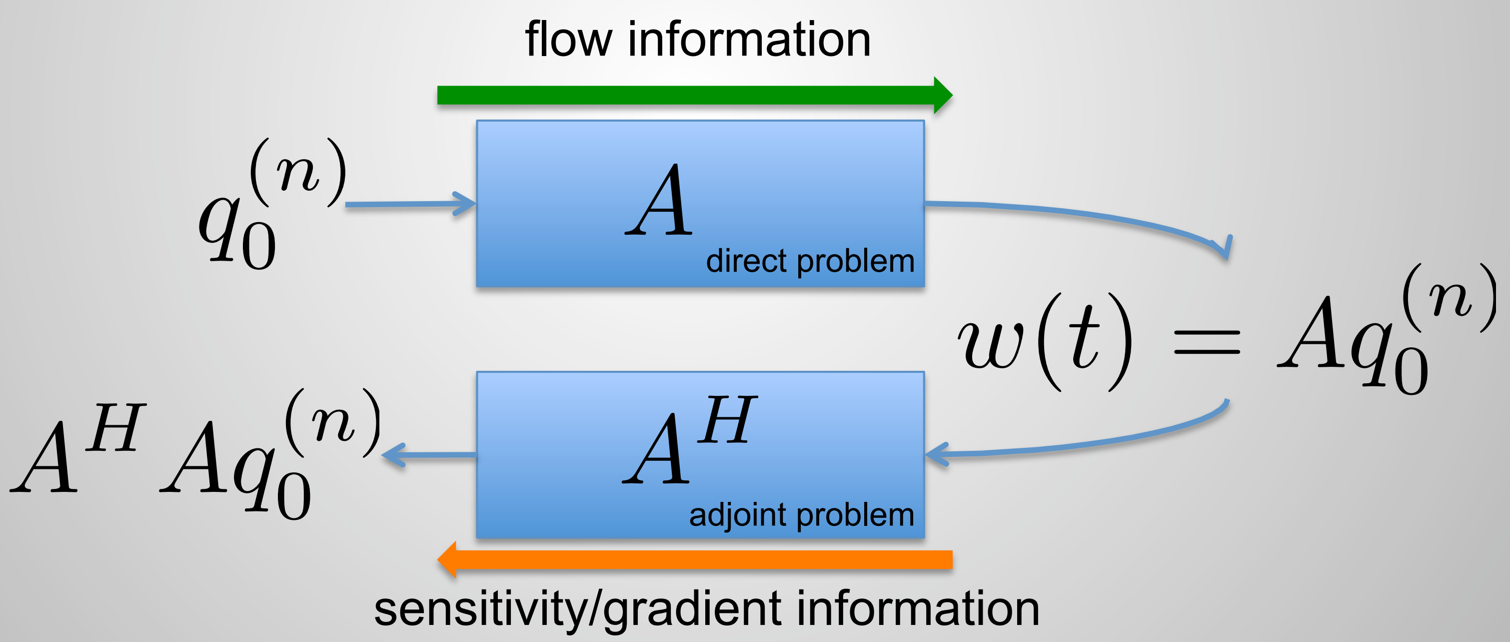
one period

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Another look at the direct-adjoint system



flow information

$q_0^{(n)}$

A

direct problem

$w(t) = Aq_0^{(n)}$

$A^H A q_0^{(n)}$

A^H

adjoint problem

sensitivity/gradient information

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

reformulate the optimal growth problem variationally

we wish to optimize

$$J = \frac{||q||^2}{||q_0||^2} \rightarrow \max$$

subject to the constraint

$$\frac{d}{dt}q - Lq = 0$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

rather than substituting the constraint directly into the cost functional …

$$J = \frac{||q||^2}{||q_0||^2} = \frac{||\exp(tL)q_0||^2}{||q_0||^2} \longrightarrow \max$$

$$\frac{d}{dt}q - Lq = 0$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow ➡

pseudo-Floquet analysis
adjoint analysis

… we enforce the equation via a Lagrange multiplier $\tilde{q}$

$$J = \frac{||q||^2}{||q_0||^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

This has the advantage that the solution to the governing equation does not have to known explicitly.

Other constraints (such as initial and boundary conditions) can be added.

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for an optimum we have to require all first variations of J to be zero

$$J = \frac{\|q\|^2}{\|q_0\|^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

$$\frac{\delta J}{\delta \tilde{q}} = 0 \quad \Rightarrow \quad \left\langle \delta\tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle = 0$$

$$\frac{\delta J}{\delta q} = 0 \quad \Rightarrow \quad \left\langle \tilde{q}, \left(\frac{d}{dt} \delta q - L\, \delta q \right) \right\rangle = 0$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for an optimum we have to require all first variations of J to be zero

$$J = \frac{||q||^2}{||q_0||^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \to \max$$

$$\frac{\delta J}{\delta \tilde{q}} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} q - Lq = 0}$$

$$\frac{\delta J}{\delta q} = 0 \quad \Rightarrow \quad \left\langle \tilde{q}, \left(\frac{d}{dt} \delta q - L \, \delta q \right) \right\rangle = 0$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for an optimum we have to require all first variations of J to be zero

$$J = \frac{||q||^2}{||q_0||^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

$$\frac{\delta J}{\delta \tilde{q}} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} q - Lq = 0}$$

$$\frac{\delta J}{\delta q} = 0 \quad \Rightarrow \quad \left\langle \left(-\frac{d}{dt} \tilde{q} - L^H \tilde{q} \right), \delta q \right\rangle = 0$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for an optimum we have to require all first variations of J to be zero

$$J = \frac{||q||^2}{||q_0||^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \to \max$$

$$\frac{\delta J}{\delta \tilde{q}} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} q - Lq = 0}$$

direct problem

$$\frac{\delta J}{\delta q} = 0 \quad \Rightarrow \quad \boxed{-\frac{d}{dt} \tilde{q} - L^H \tilde{q} = 0}$$

adjoint problem

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

adjoint variables can be interpreted as sensitivities

$$J = \mathrm{obj} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

let us add an external body force to the governing equations

$$\frac{d}{dt} q - Lq = \enclose{circle}{f}$$

external force

$$\delta J = -\langle \tilde{q}, \delta f \rangle$$

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

adjoint variables can be interpreted as sensitivities

$$J = \mathrm{obj} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

let us add an external body force to the governing equations

$$\frac{d}{dt} q - Lq = \textcircled{f}$$

external force

$$\boxed{\nabla_f J = -\tilde{q}}$$

sensitivity to external body force

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: which adjoint variable measures the sensitivity to a mass source/sink?

$$J = \mathrm{obj} - \underbrace{\langle \tilde{\mathbf{u}}, NS(\mathbf{u}) \rangle}_{\text{enforcing momentum conservation}} - \underbrace{\langle \xi, \nabla \cdot \mathbf{u} \rangle}_{\text{enforcing mass conservation}}$$

$$-\langle \xi, \nabla \cdot \mathbf{u} \rangle \quad \xrightarrow[\text{by parts}]{\text{integration}} \quad \langle \nabla \xi, \delta \mathbf{u} \rangle$$

ξ is the adjoint pressure

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow →

pseudo-Floquet analysis
adjoint analysis

Example: which adjoint variable measures the sensitivity to a mass source/sink?

$$J = \mathrm{obj} - \langle \tilde{\mathbf{u}}, NS(\mathbf{u}) \rangle - \langle \xi, \nabla \cdot \mathbf{u} \rangle$$

enforcing momentum conservation

enforcing mass conservation

assuming a mass source/sink

$$\nabla \cdot \mathbf{u} = Q$$

$$\delta J = \langle \xi, \delta Q \rangle$$

adjoint pressure =
sensitivity to a mass source/sink

Generalizations of nonmodal stability analysis

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for the incompressible Navier-Stokes equations

← forcing | sensitivity →

$$\frac{\partial \mathbf{u}}{\partial t} + \mathrm{advdiff}(\mathbf{U}, \mathbf{u}) + \nabla p = \mathbf{F}$$

$$\nabla \cdot \mathbf{u} = Q$$

$$\mathbf{u} = \mathbf{u}_w \quad \mathrm{on} \quad y = 0$$

$$\nabla_{\mathbf{F}} J = \tilde{\mathbf{u}}$$

$$\nabla_Q J = \tilde{p}$$

$$\nabla_{\mathbf{u}_w} J = \tilde{\sigma}\big|_w$$

Generalizations of nonmodal stability analysis

nonlinear perturbation dynamics → adjoint analysis
with check-pointing

the variational formulation also allows us to add ***nonlinear*** constraints to the cost functional

$$J = \mathrm{obj} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - N(q) \right) \right\rangle \rightarrow \max$$

nonlinear Navier-Stokes equations

How does this affect the adjoint looping ?

Generalizations of nonmodal stability analysis

nonlinear perturbation dynamics → adjoint analysis with check-pointing

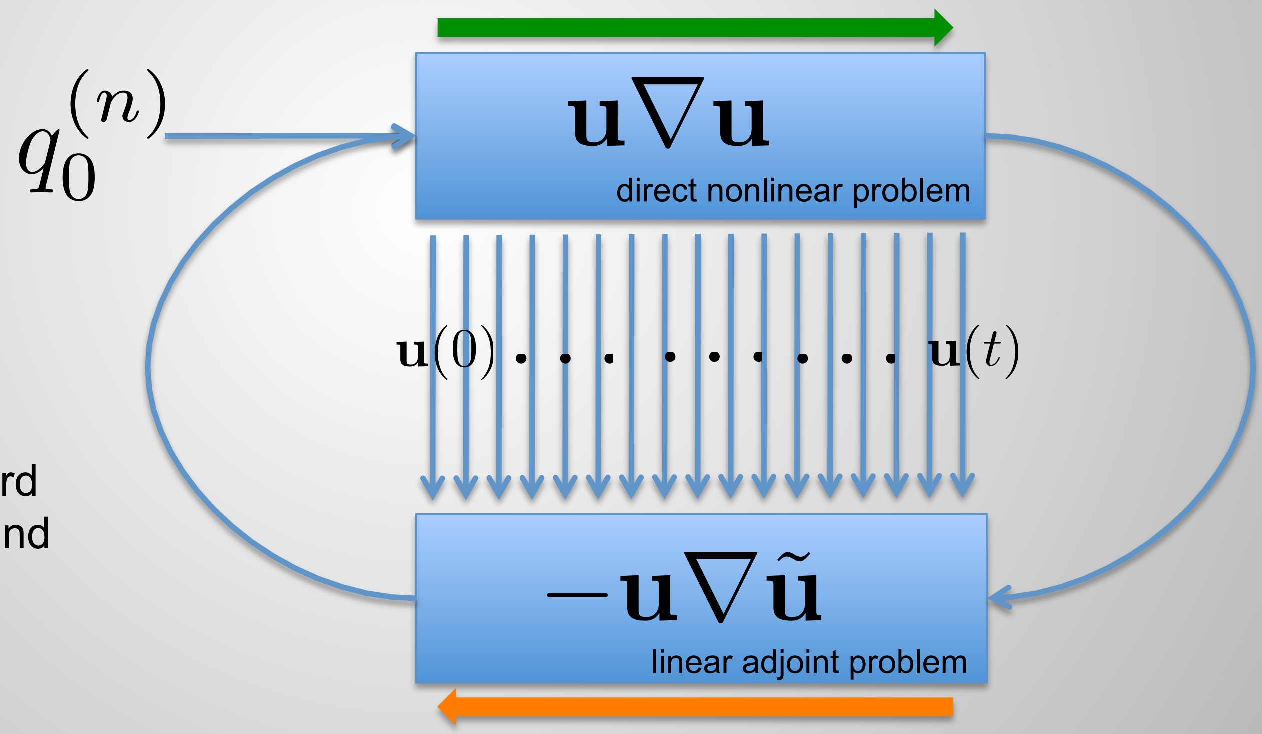
Example: nonlinear advective terms

$$\langle \tilde{\mathbf{u}}, \mathbf{u}\nabla\mathbf{u}\rangle \xrightarrow{\text{first variation}} \langle -\enclose{circle}{\mathbf{u}}\nabla\tilde{\mathbf{u}}, \delta\mathbf{u}\rangle$$

We have **direct** terms appearing in the **adjoint** equation.

Generalizations of nonmodal stability analysis

nonlinear perturbation dynamics → adjoint analysis with check-pointing



checkpointing

the flow fields at the forward sweep have to be saved and injected into the backward sweep

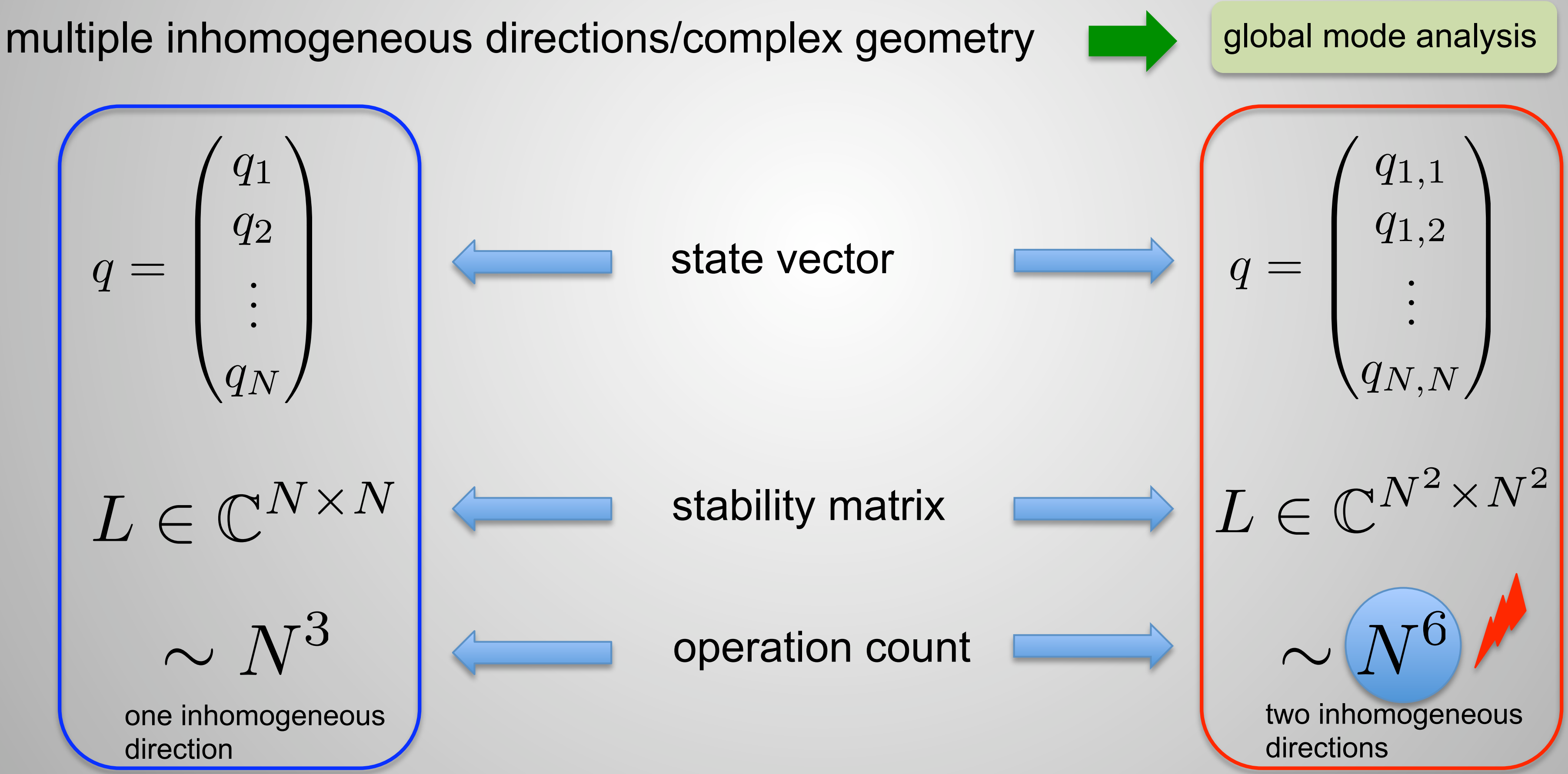
Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

for most industrial applications we cannot assume the existence of homogeneous directions that can be treated by a Fourier transform

rather, the eigenfunction will depend on more than one inhomogeneous coordinate direction

Generalizations of nonmodal stability analysis



multiple inhomogeneous directions/complex geometry → global mode analysis

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}$$

state vector

$$q = \begin{pmatrix} q_{1,1} \\ q_{1,2} \\ \vdots \\ q_{N,N} \end{pmatrix}$$

$$L \in \mathbb{C}^{N \times N}$$

stability matrix

$$L \in \mathbb{C}^{N^2 \times N^2}$$

$$\sim N^3$$

one inhomogeneous direction

operation count

$$\sim N^6$$

two inhomogeneous directions

Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

→ direct eigenvalue algorithms quickly become prohibitively expensive

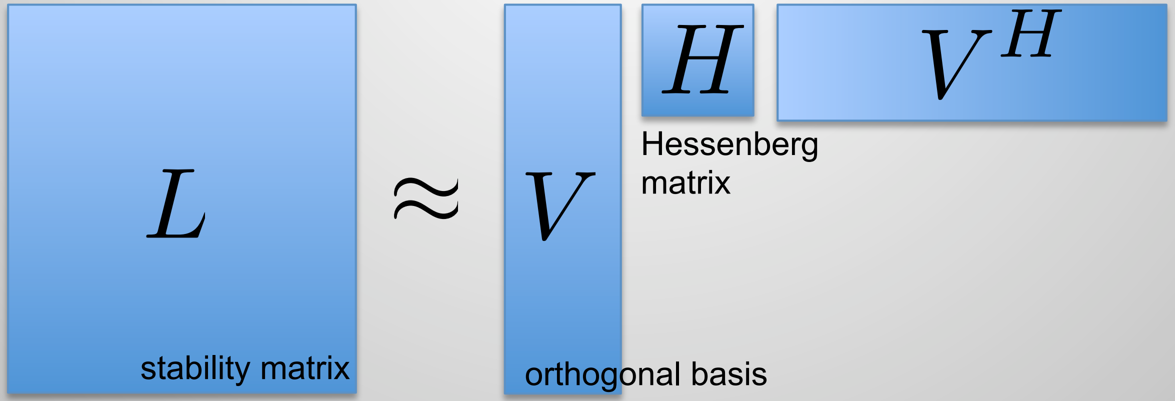
→ iterative eigenvalue algorithms (Arnoldi technique) have to be used

Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

Arnoldi algorithm

represent the (large) stability matrix by a low-rank approximation based on an orthogonal basis



$$L \approx V \; H \; V^H$$

Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

```
$q_k = L \; q_{k-1}$
for $j = 1 : k-1$
    $H_{j,k-1} = \langle q_j , q_k \rangle$
    $q_k = q_k - H_{j,k-1} \; q_j$
end
$H_{k,k-1} = \|q_k\|$
$q_k = q_k / H_{k,k-1}$
```

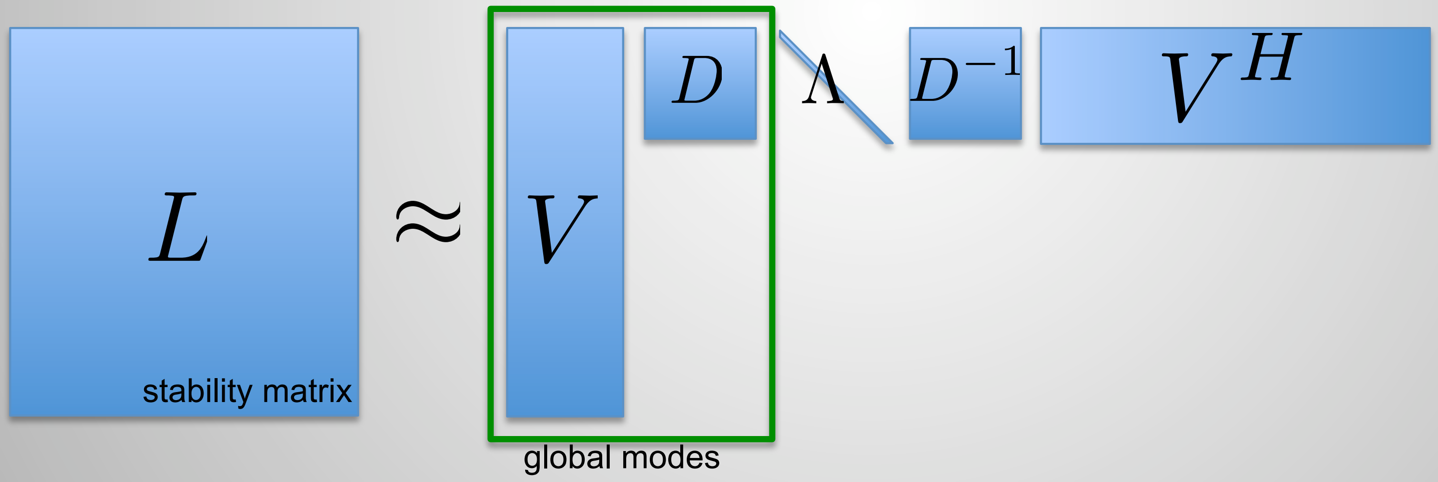
only multiplications by L are necessary

→ $$\mathrm{eig}\{L\} \approx \mathrm{eig}\{H\}$$

Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

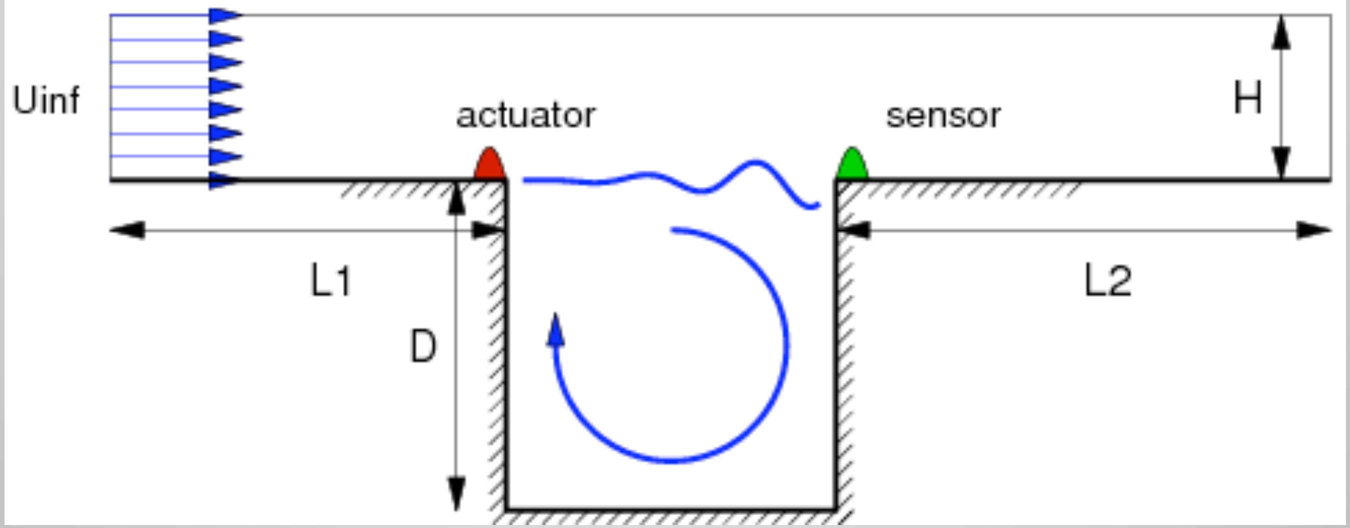
computing global modes by diagonalizing $H = D\Lambda D^{-1}$



Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

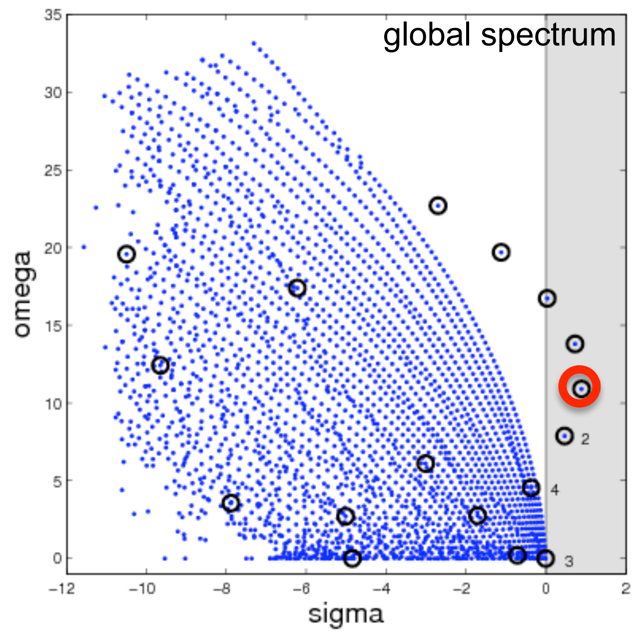
Examples of global modes: open cavity flow (two-dimensional)



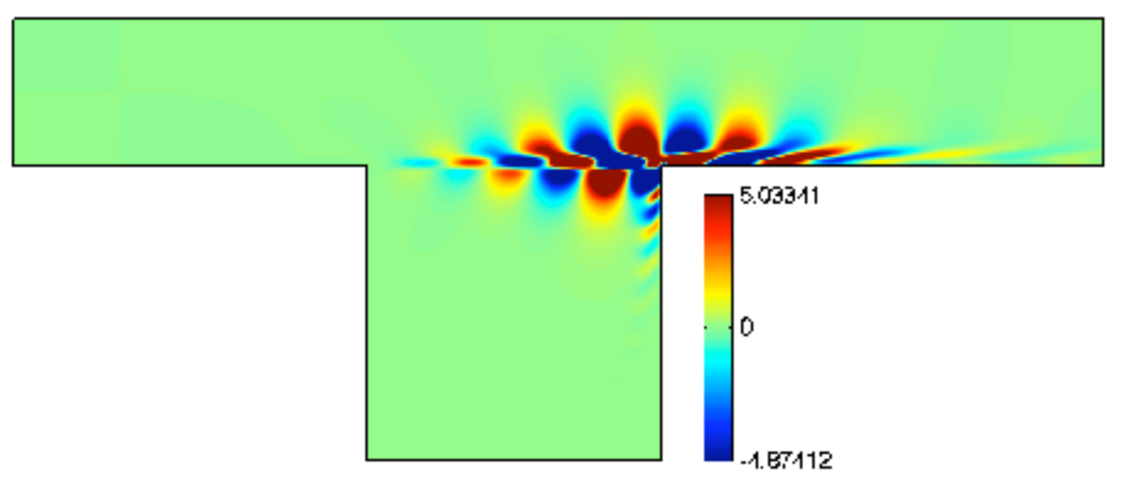
Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → **global mode analysis**

Examples of global modes: open cavity flow (two-dimensional)



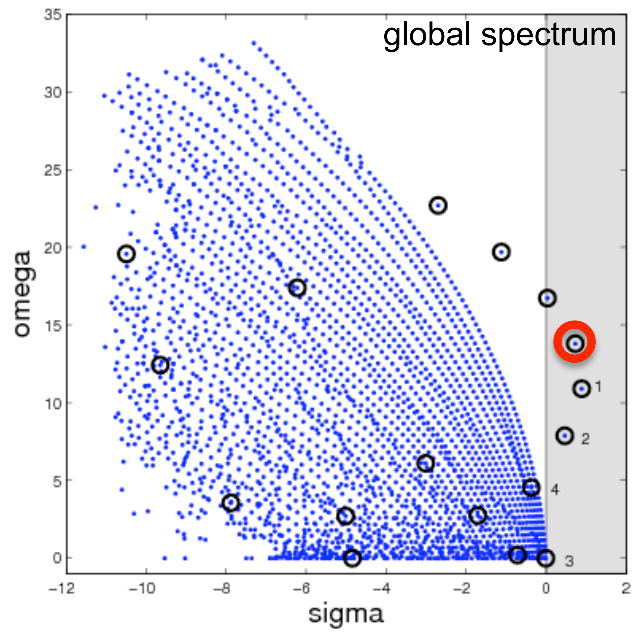
global mode



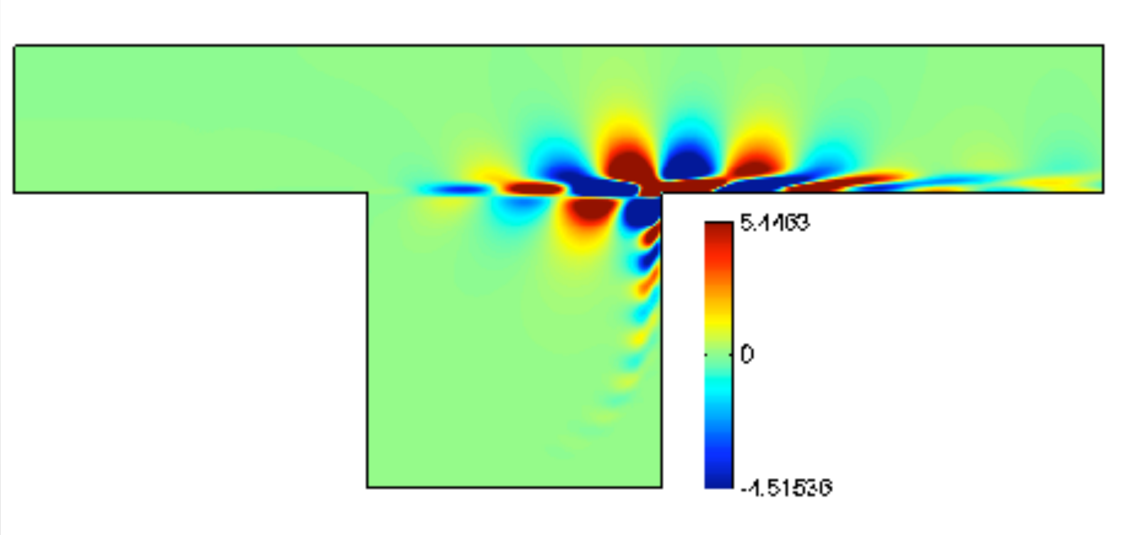
Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: open cavity flow (two-dimensional)



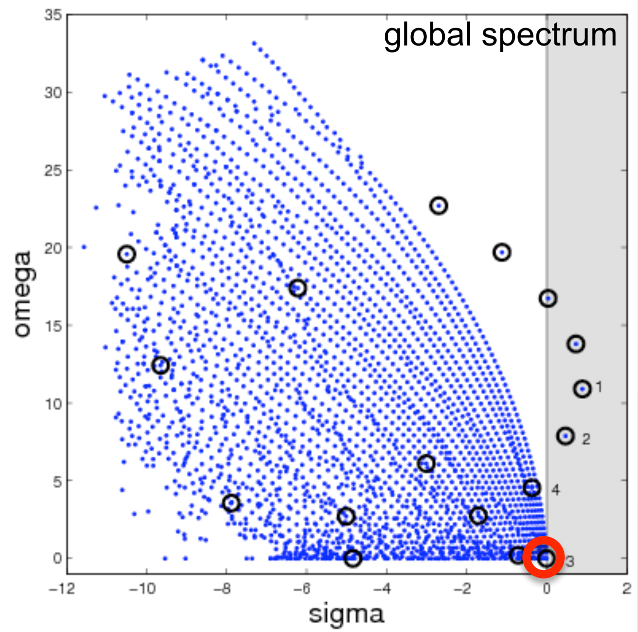
global mode

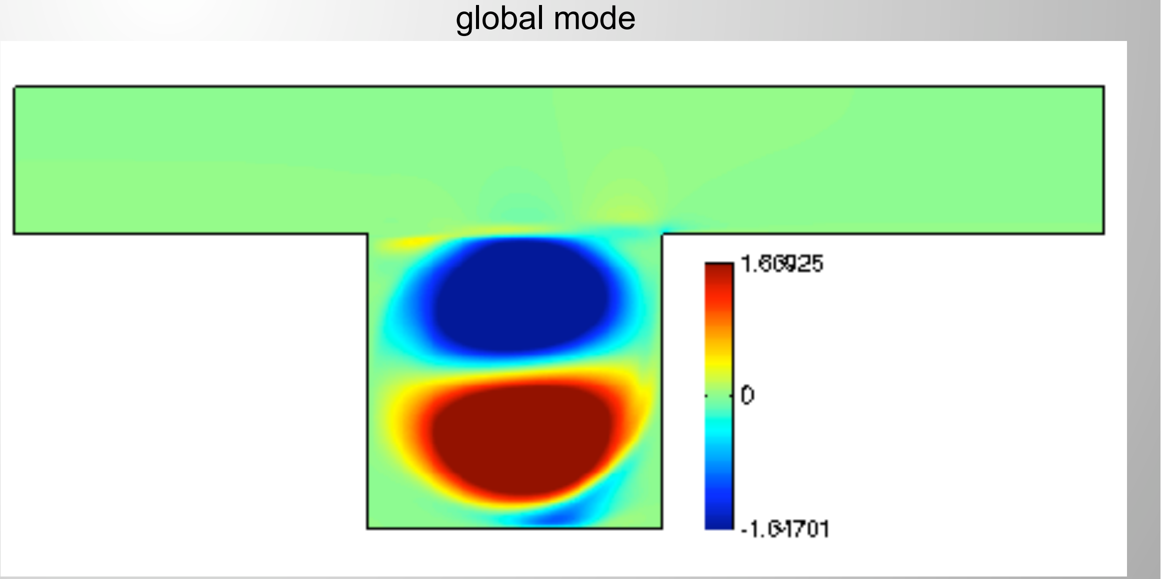


Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry ➡ global mode analysis

Examples of global modes: open cavity flow (two-dimensional)

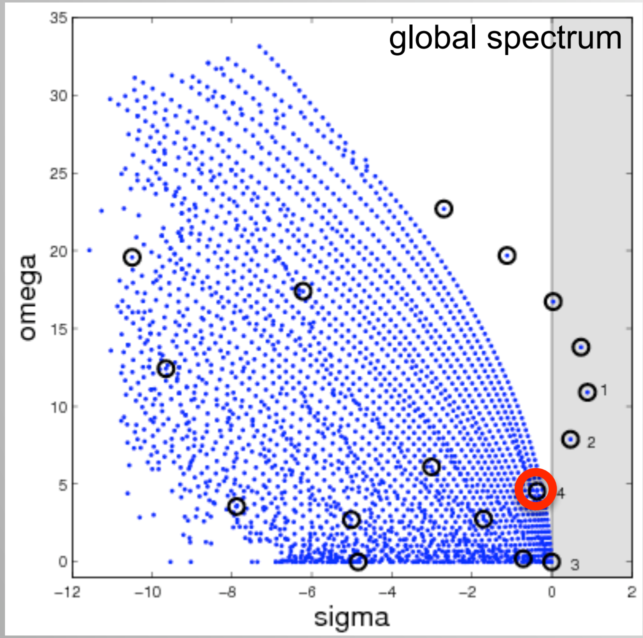




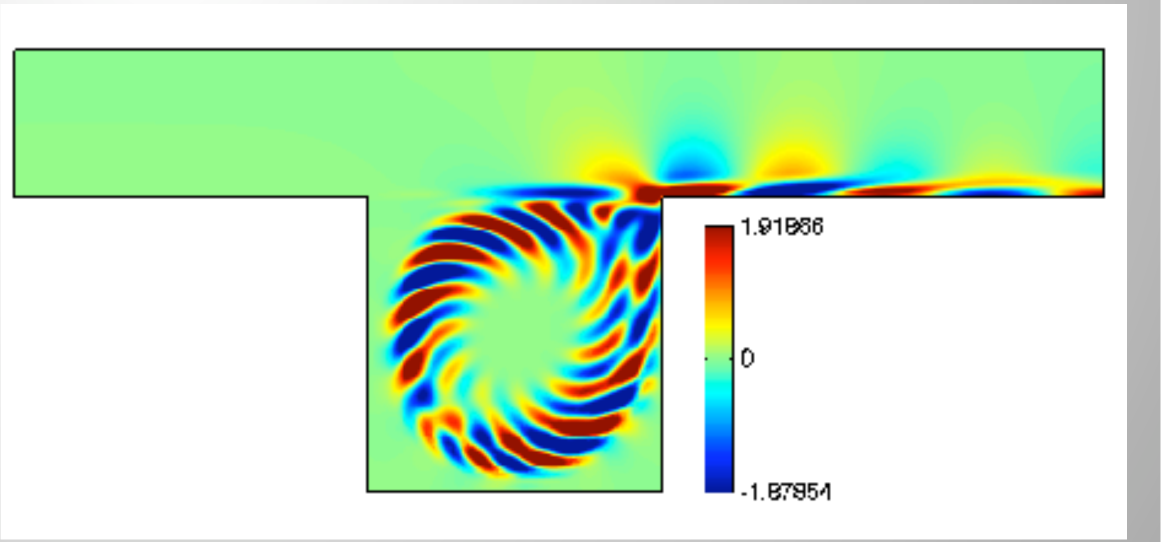
Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: open cavity flow (two-dimensional)



global mode



Generalizations of nonmodal stability analysis

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: jet in cross flow (three-dimensional)