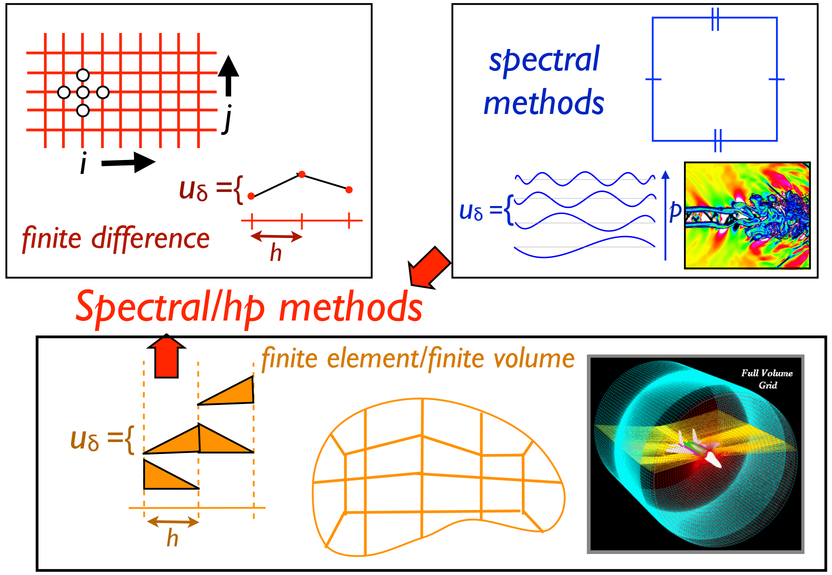
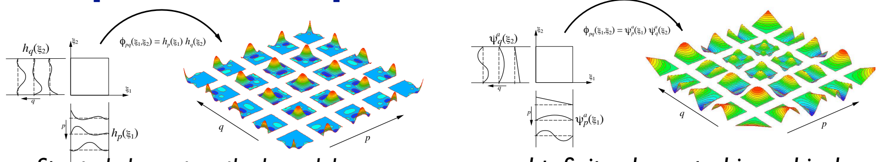
Numerical Methods for Linear Stability *Beyond Matlab*

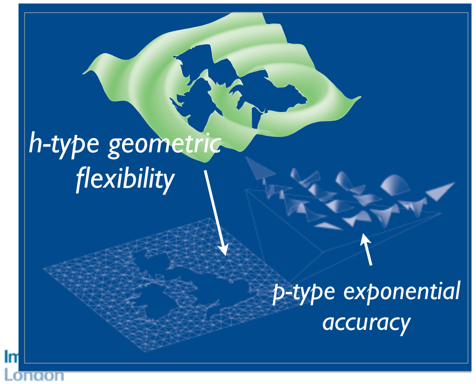
- Spectral/hp element methods
- Nektar++ code
- Linearised Stability Analysis

Solution Techniques



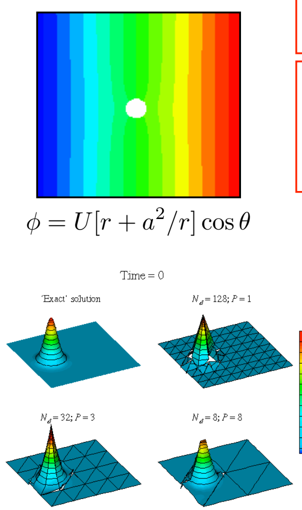
Spectral/hp element method



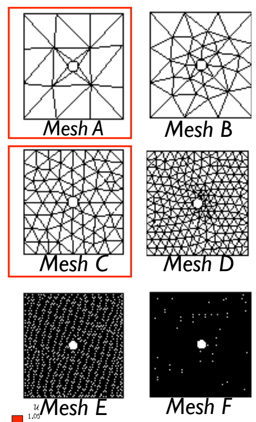


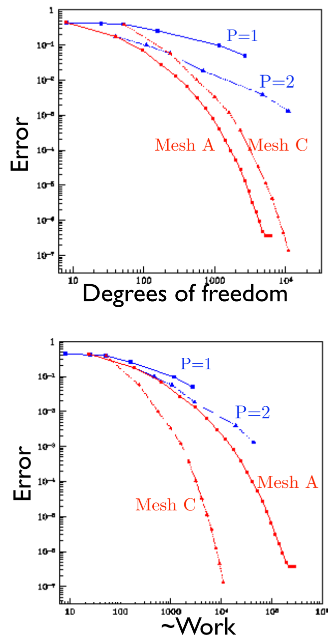
$$\text{Work} \approx C(h)p^{(Dim+1)}$$
$$\text{Error} \approx K(u,p)\, h^p$$

Why Spectral/hp elements methods?



$\phi = U[r + a^2/r]\cos\theta$





Books

Nektar++ *Overview: What is it?*

S.J. Sherwin, R. M. Kirby

Chris Cantwell, Gabrielle Rocco

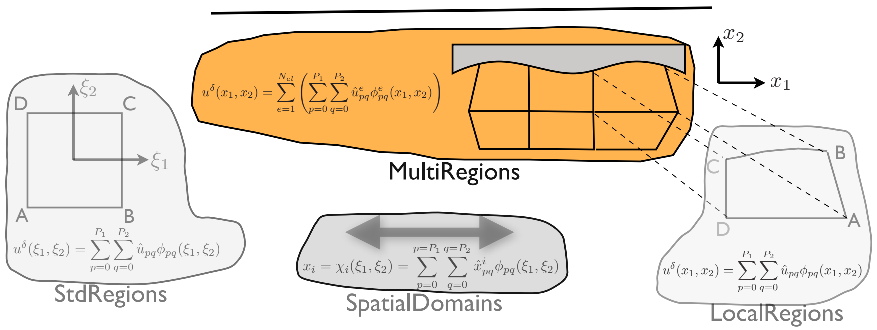
Nektar++ is an open source software library currently being developed and designed to provide a bridge to the community – to provide a toolbox of data structures and algorithms which implement the spectral/hp element method, a high-order numerical method yielding fast error convergence. It is implemented as a C++ object-oriented toolkit which allows developers to implement spectral element solvers for a variety of different engineering problems.

For more information, go to: *www.nektar.info*

Libraries to operator mapping

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

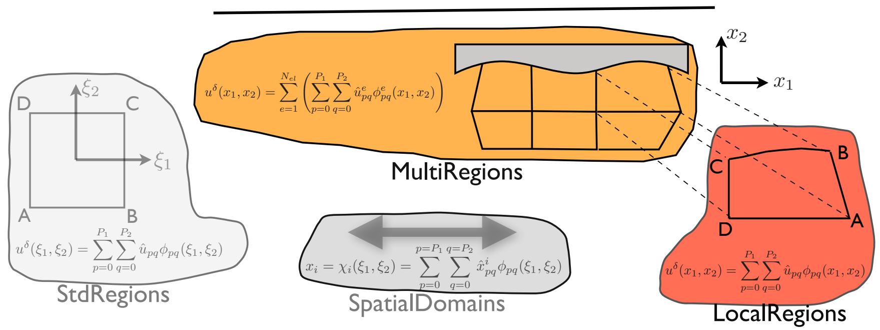
$$\mathbf{f}[i] = \int \Phi_i f^* dx$$



Libraries to operator mapping

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(x) f^* dx$$



Libraries to operator mapping

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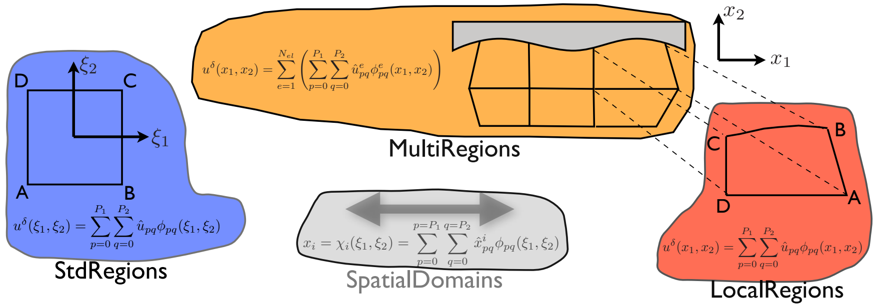

Libraries to operator mapping

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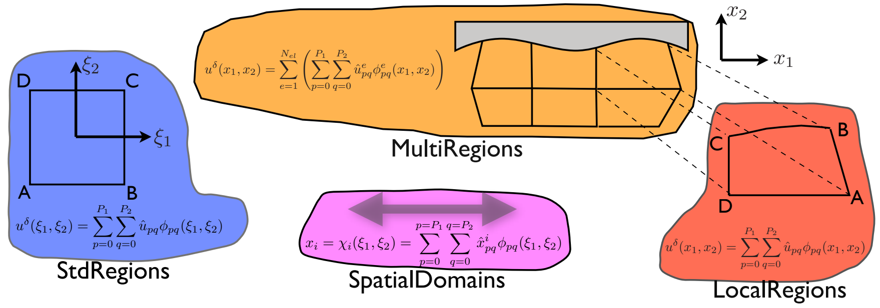
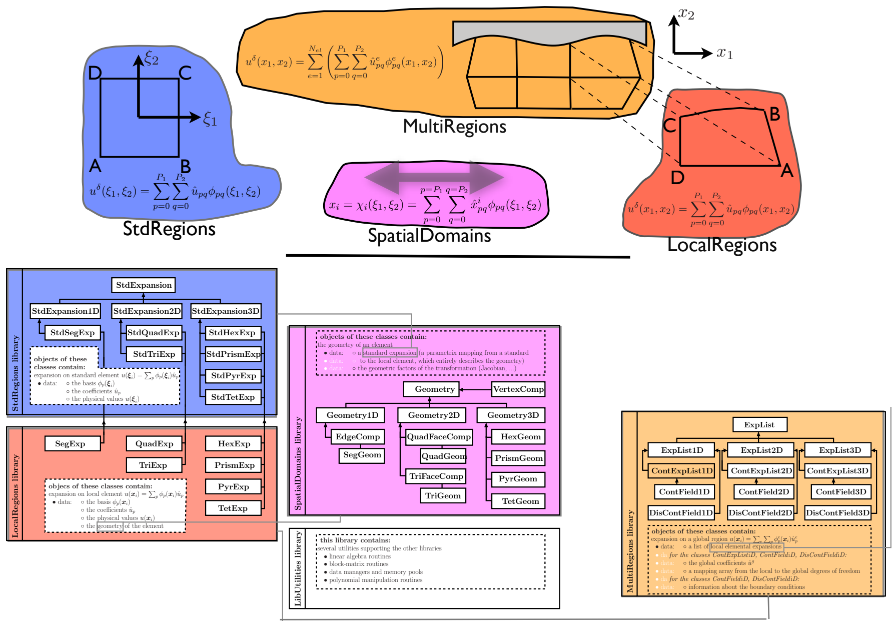


Nektar++ code



Linear stability analysis

- Eigenvalues from inverse linearised operator
- Time steppers approach to linearised eigenvalues
- Transient growth of linearised operator

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Linearised Navier-Stokes

$\boldsymbol{u} = \boldsymbol{U} + \epsilon \boldsymbol{u}'$ — *base flow + perturbation*

$\partial_t \boldsymbol{u}' = \partial_{\boldsymbol{U}} \boldsymbol{N}(\boldsymbol{u}') + \boldsymbol{L}(\boldsymbol{u}')$ — *keep terms linear in ϵ*

$\boldsymbol{u}' \cdot \nabla \boldsymbol{U} + \boldsymbol{U} \cdot \nabla \boldsymbol{u}'$ — *non-symmetric*

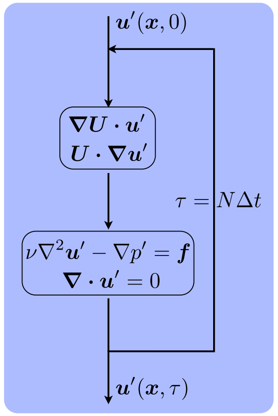
Linear system:

$$\mathbf{u'}_t = \mathcal{L}(U,t)\mathbf{u}' \qquad (\nabla \cdot \mathbf{u}' = 0)$$

Integrate in time:

$$\mathbf{u}'(\mathbf{x},\tau) = \mathcal{A}(U,\tau)\mathbf{u}'(\mathbf{x},0) \iff$$

$$\mathcal{A}(U,\tau) = \exp\left(\int_0^\tau \mathcal{L}(U,t)dt\right)$$



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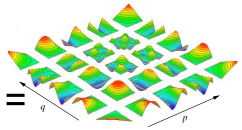
Direct inversion of linearised Navier-Stokes system

(Ainsworth & Sherwin, *CMAME*, 1999 / Sherwin & Ainsworth *APNUM 2000)*

$$\partial_t \boldsymbol{u}' = \mathcal{L}(\boldsymbol{U},t)\boldsymbol{u}'$$

$$\mathcal{L}(\boldsymbol{U},t)\boldsymbol{u}' = \begin{bmatrix} A & D_{bnd}^T & B \\ D_{bnd} & 0 & D_{int}^T \\ \tilde{B}^T & D_{int} & C \end{bmatrix} \begin{bmatrix} \mathbf{v}_{bnd} \\ p \\ \mathbf{v}_{\mathbf{int}} \end{bmatrix}$$

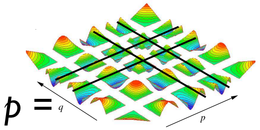
v =



$$A[n,m] = (\nabla\phi_n^b, \nu\nabla\phi_m^b) + (\phi_n^b, \mathbf{U}\cdot\nabla\phi_m^b) + (\phi_n^b \nabla^T \mathbf{U}\phi_m^b)$$

$$D_{bnd}[n,m] = (\psi_m, \nabla\phi^b)$$

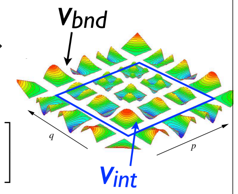
p =



Static condensation:

$$\begin{bmatrix} I & 0 & -BC^{-1} \\ 0 & I & D_{int}^T C^{-1} \\ 0 & 0 & I \end{bmatrix} \left\{ \begin{bmatrix} A & D_{bnd}^T & B \\ D_{bnd} & 0 & D_{int}^T \\ \tilde{B}^T & D_{int} & C \end{bmatrix} \begin{bmatrix} \mathbf{v}_{bnd} \\ p \\ \mathbf{v}_{\mathbf{int}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{bnd} \\ 0 \\ \mathbf{f}_{\mathbf{int}} \end{bmatrix} \right\}$$

$$\begin{bmatrix} A - BC^{-1}\tilde{B}^T & D_{bnd}^T - BC^{-1}D_{int} & 0 \\ D_{bnd} - D_{int}^T C^{-1}\tilde{B}^T & -D_{int}^T C^{-1} D_{int} & 0 \\ \tilde{B}^T & D_{int} & C \end{bmatrix} \begin{bmatrix} \mathbf{v}_{bnd} \\ p \\ \mathbf{v}_{\mathbf{int}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{bnd} - BC^{-1}\mathbf{f}_{int} \\ f_p = -D_{int}^T C^{-1}\mathbf{f}_{int} \\ \mathbf{f}_{\mathbf{int}} \end{bmatrix}$$



Smallest eigenvalues of *L is largest eigenvalue of* L^{-1}

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Restarted Eigenvalue algorithm (*IJNMF 08)*

ARPACK: http://www.caam.rice.edu/software/ARPACK/

$$\mathbf{A}\mathbf{Q}_k = \mathbf{Q}_k\mathbf{H}_k + h_{k,k-1}\mathbf{v}_k\mathbf{e}_{k-1}^H.$$

$$\lambda_k(\mathbf{A}) \approx \lambda_{\max}(\mathbf{H}_k) \qquad \mathbf{w}_{\max} \approx \mathbf{Q}_k\,\mathbf{y}_{\max}$$

1. Generate a Krylov subspace *T*

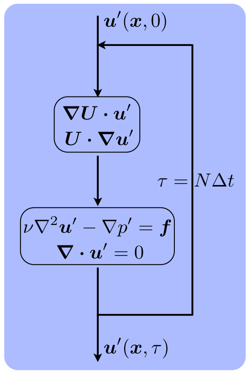
$$T = \{\boldsymbol{u}', \mathcal{A}\boldsymbol{u}_0', \mathcal{A}^2\boldsymbol{u}_0', \ldots, \mathcal{A}^{k-1}\boldsymbol{u}_0'\}$$

2. QR factorize *T and* Calculate Hessenberg matrix *H*:

$$h_{i,j} = \frac{1}{r_{j,j}}\left(r_{i,j+1} - \sum_{l=0}^{j-1} h_{i,l}r_{l,j}\right)$$

3. Use LAPACK to calculate eigensystem of *H*

$\mathcal{A}\boldsymbol{u}_0 \equiv$



In practice use a fixed K_{max}.
If iteration exceeds K_{max}, then discard oldest vector , i.e. restart with second oldest vector.

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Timestepper stability analysis

Tuckerman & Barkley (2000), Barkley & Henderson (1996)

$$\partial_t \boldsymbol{u}' = \mathcal{L}(\boldsymbol{U},t)\boldsymbol{u}' \qquad \boldsymbol{u}'(\boldsymbol{x},\tau) = \mathcal{A}(\boldsymbol{U},\tau)\boldsymbol{u}'(\boldsymbol{x},0)$$

Asymptotic instability if:

$$\mathrm{Re}(\lambda_i(\mathcal{L})) > 0 \implies \boldsymbol{u}' \propto \mathrm{e}^{\lambda_i t}$$

equivalently

$$|\mu_i(\mathcal{A})| > 1 \implies \boldsymbol{u}'(\boldsymbol{x},\tau) > \boldsymbol{u}'(\boldsymbol{x},0)$$

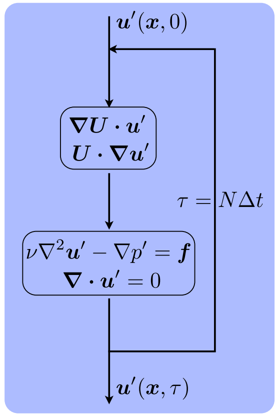
where

$$\mu_i = \mathrm{e}^{\lambda_i \tau}$$

Note: $\max_i \mu_i \Rightarrow \max_i Re(\lambda_i)$

If $\boldsymbol{U}(t)$ is periodic, μ_i are Floquet multipliers

$\mathcal{A}(\boldsymbol{U},\tau)\boldsymbol{u}_0' \equiv$



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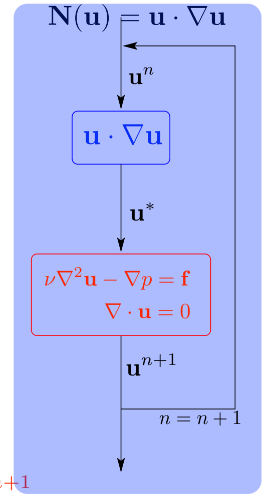
Navier-Stokes Time-stepping

Navier–Stokes: $$\partial_t \mathbf{u} + \mathbf{N}(\mathbf{u}) = -\nabla p + \nu \nabla^2 \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$



Velocity correction scheme *(aka stiffly stable):*

Orszag, Israeli, Deville (90), Karnaidakis Israeli, Orszag (1991), Guermond & Shen (2003)

Advection: $$u^* = -\sum_{q=1}^{J} \alpha_q \mathbf{u}^{n-q} - \Delta t \sum_{q=0}^{J-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q})$$

Pressure Poisson: $$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \mathbf{u}^*$$

Helmholtz: $$\nabla^2 \mathbf{u}^{n+1} - \frac{\alpha_0}{\nu \Delta t} \mathbf{u}^{n+1} = -\frac{\mathbf{u}^*}{\nu \Delta t} + \frac{1}{\nu} \nabla p^{n+1}$$

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Transient growth: optimal perturbations

Energy of perturbations over time τ

$$G(\tau) = \frac{E(\tau)}{E(0)} = \frac{(\boldsymbol{u}'(\tau), \boldsymbol{u}'(\tau))}{(\boldsymbol{u}'(0), \boldsymbol{u}'(0))}$$

Recall: $\boldsymbol{u}'(\boldsymbol{x}, \tau) = \mathcal{A}(\boldsymbol{U}, \tau)\boldsymbol{u}'(\boldsymbol{x}, 0)$

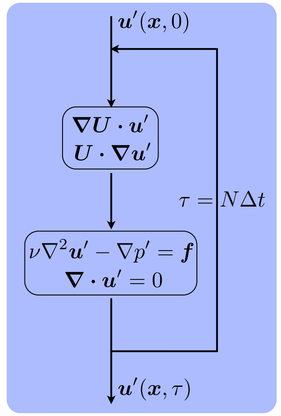
$$G(\tau) = \frac{(\mathcal{A}\boldsymbol{u}'(0), \mathcal{A}\boldsymbol{u}'(0))}{(\boldsymbol{u}'(0), \boldsymbol{u}'(0))} = \frac{(\boldsymbol{u}'(0), \mathcal{A}^*\mathcal{A}\boldsymbol{u}'(0))}{(\boldsymbol{u}'(0), \boldsymbol{u}'(0))}$$

Equivalent to eigenvalue problem:

$$(\boldsymbol{u}'(0), \{\mathcal{A}^*\mathcal{A}\boldsymbol{u}'(0) = G(\tau)\boldsymbol{u}'(0)\})$$

$$\mathcal{A}^*\mathcal{A}\boldsymbol{u}'(0) - G(\tau)\boldsymbol{u}'(0) = 0$$

$\mathcal{A}(\boldsymbol{U}, \tau)\boldsymbol{u}'_0 \equiv$



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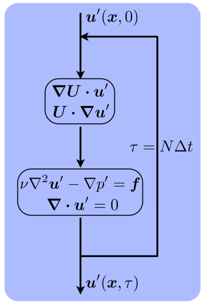
Transient growth: optimal perturbations

$$G(\tau) = \max_{\boldsymbol{u}'(0)} \frac{E(\tau)}{E(0)} \implies \lambda_{\max}(\mathcal{A}^*\mathcal{A}) \quad \Rightarrow \lambda_{\max} = [SVD_{\max}(\mathcal{A})]^2$$

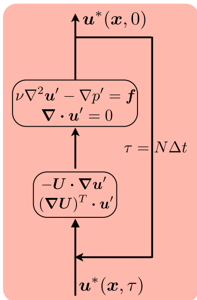
$$\mathcal{H} \equiv \begin{cases} \left[\begin{array}{c|c} -\partial_t - \mathrm{DN} + Re^{-1}\nabla^2 & -\nabla \\ \hline \nabla\cdot & 0 \end{array}\right] \\ \mathrm{DN}\,\boldsymbol{u}' = (\boldsymbol{U}\cdot\nabla)\boldsymbol{u}' + (\nabla\boldsymbol{U})\cdot\boldsymbol{u}' \end{cases}$$

$$\mathcal{H}^* \equiv \begin{cases} \left[\begin{array}{c|c} \partial_t - \mathrm{DN}^* + Re^{-1}\nabla^2 & -\nabla \\ \hline \nabla\cdot & 0 \end{array}\right] \\ \mathrm{DN}^*\boldsymbol{u}^* = -(\boldsymbol{U}\cdot\nabla)\boldsymbol{u}^* + (\nabla\boldsymbol{U})^{\mathrm{T}}\cdot\boldsymbol{u}^* \end{cases}$$

$\boldsymbol{u}^* = \mathcal{A}\boldsymbol{u}'$



$\mathcal{A}^*\boldsymbol{u}^*$



Direct optimal growth analysis for timesteppers, Barkley, D, Blackburn, H.M, Sherwin, S.J , International Journal for Numerical Methods in Fluids, Vol: 57, pages: 1435-145 , 2008

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Direct inversion vs. Time stepping.

	Direct invert	Time stepping
Linear eigenvalue steady base flow	Yes (quick) might need shifting	Yes (slow) Exp. method finds leading eigenvalue
Adjoint eigenvalue steady base flow	Yes (quick) might need shifting	Yes (slow) Exp. method finds leading eigenvalue
Linear Floquet mode about unsteady base flow	No	Yes
Transient growth	No	Yes

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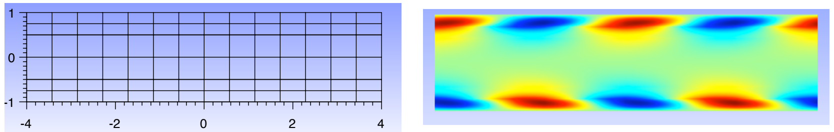
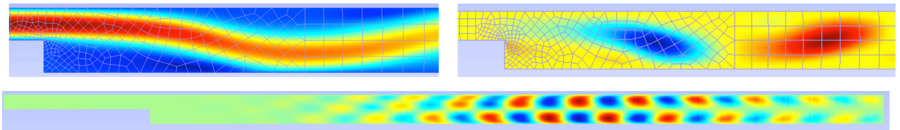
Test cases

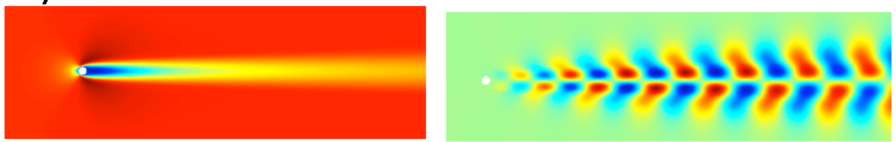
Channel flow:



Backward facing step:



Cylinder flow:



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A few papers

Numerical methods:
- *Direct optimal growth analysis for timesteppers, Barkley, D, Blackburn, H.M, Sherwin, S.J , International Journal for Numerical Methods in Fluids, Vol: 57, 1435-145 , 2008*

Cylinder flow papers:
- *Secondary instabilities in the flow around two circular cylinders in tandem*, Carmo, B.S., Meneghini, J. Sherwin, S.J., J. Fluid Mech. Vol 644,395-431 , 2010
- *Wake transition in the flow around two circular cylinders in staggered arrangements,* Carmo, B.S, Sherwin, S.J, Bearman, P.W, Willden, R.H.J , Journal of Fluid Mechanics, Vol: 597, 1-29 , 2008

Backward facing step transient growth:
- *Convective instability and transient growth in flow over a backwards-facing step,* Blackburn, H.M, Barkley, D, Sherwin, S.J , Journal of Fluid Mechanics, Vol: 603, 271-304 , 2008

Stenotic flow:
- *Convective instability and transient growth in steady and pulsatile stenotic flows,* Blackburn, H.M, Sherwin, S.J, Barkley, D , Journal of Fluid Mechanics, Vol: 607, 267-277 , 2008
- *Instability modes and transition of pulsatile stenotic flow: pulse-period dependence,* Blackburn, H.M, Sherwin, S.J , Journal of Fluid Mechanics, Vol: 573, 57 , 2007
- *Three-dimensional instabilities and transition of steady and pulsatile axisymmetric stenotic flows*, Sherwin, SJ, Blackburn, HM, , J. Fluid Mechanics, Vol: 533, 297 - 327 , 2005

Low pressure turbine flow:
- *Transient growth mechanisms of low Reynolds number flow over a low pressure turbine blade*, A.S. Sharma, N. Abdessemed, S.J.Sherwin, V. Theofilis, Theo & Comp Fluid Dyn. 25, 19-30 20011
- *Linear instability analysis of low pressure turbine flows*, N. Abdessemed, S.J. Sherwin, and V. Theofilis, J. Fluid Mech. , 628, 57-83, 2009.

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