4. Weakly nonlinear theory

- This describes the next stages of disturbance evolution when amplitudes start to become too large for linear theory to apply.
- Nonlinearity might be stabilizing, and could lead to a saturated nonlinear equilibrium state of finite amplitude:
• Or nonlinearity might be destabilizing, and could lead to threshold behaviour:

• These roles of nonlinearity, and others, can be determined by using perturbation theory, i.e. weakly nonlinear theory.

• This requires both small amplitudes and weak growth rates.
Solvability conditions

- When we include nonlinear effects we will encounter disturbance equations with right hand sides, e.g. forced Orr-Sommerfeld equations.
- Forced equations do not necessarily have solutions satisfying homogeneous boundary conditions.
- The solvability of such equations is crucial to the development of weakly nonlinear theory.
- The essence of the matter can be illustrated using this simple ODE:

\[ \frac{d^2 y}{dx^2} + \omega^2 y = \sin \Omega x, \quad y(0) = 0, \quad y(L) = 0 \quad (300) \]

where \( \omega \neq \pm \Omega \).
• The general solution of (300) is

\[ y = A \cos \omega x + B \sin \omega x + \frac{1}{\omega^2 - \Omega^2} \sin \Omega x. \]  

(301)

• Applying the boundary conditions to (301) gives

\[ 0 = A \]

(302)

\[ 0 = B \sin \omega L + \frac{1}{\omega^2 - \Omega^2} \sin \Omega L. \]

(303)

• If \( \sin \omega L \neq 0 \), then (303) has a solution:

\[ B = -\frac{\sin \Omega L}{(\omega^2 - \Omega^2) \sin \omega L}. \]

(304)
• However, if \( \sin \omega L = 0 \), then \( \omega \) is an eigenvalue of the homogeneous (unforced) problem, and (303) may have no solution.

• Therefore, a solution of the forced problem exists if \( \omega \) is not an eigenvalue.

• If \( \omega \) is an eigenvalue, then a solution to (303) only exists if

\[
\sin \Omega L = 0. \tag{305}
\]

• (305) is the solvability condition required for (300) when \( \omega \) is an eigenvalue.
Adjoint equations

- The solvability condition can be obtained even when we do not have the general solution.
- Consider

\[ y'' + \omega^2 y = f(x), \quad y(0) = 0, \quad y(L) = 0 \]  \hspace{1cm} (306)

where \( \omega \) is an eigenvalue of the homogeneous problem.

- What is the solvability condition on \( f \)?

- Multiply (306) by a function \( g(x) \), which satisfies the same homogeneous boundary conditions as \( y \), i.e. \( g(0) = 0 \) and \( g(L) = 0 \), and integrate from \( x = 0 \) to \( x = L \):

\[
\int_0^L g \left( y'' + \omega^2 y \right) \, dx = \int_0^L gf \, dx. \]  \hspace{1cm} (307)
• Integrate by parts using the boundary conditions on \( y \) and \( g \):

\[
\int_0^L gy'' \, dx = [gy']_0^L - \int_0^L g'y' \, dx
\]

\[
= -[g'y]_0^L + \int_0^L g''y \, dx
\]

\[
= \int_0^L g''y \, dx. \tag{308}
\]

• Substitute (308) into (307):

\[
\int_0^L y (g'' + \omega^2 g) \, dx = \int_0^L gf \, dx. \tag{309}
\]
Choose $g$ to satisfy

$$g'' + \omega^2 g = 0. \quad (310)$$

The equation for $g$ obtained in this way, (310), is called the adjoint equation.

This example happens to be self-adjoint (the adjoint equation here is the same as the original equation).

Substituting (310) into (309) gives the solvability condition for (306):

$$\int_0^L g f \, dx = 0. \quad (311)$$
• In the previous example, \( f(x) = \sin \Omega x \), and \( g = \sin \omega x \) satisfies the adjoint equation (310) and \( g(0) = g(L) = 0 \) since \( \omega = n\pi/L \) is an eigenvalue.

• Substituting these \( f \) and \( g \) into the solvability condition (311) gives

\[
0 = \int_0^L \sin \omega x \sin \Omega x \, dx
\]

\[
= \int_0^L \frac{1}{2} [\cos(\omega - \Omega)x - \cos(\omega + \Omega)x] \, dx
\]

\[
= \frac{\omega}{\Omega^2 - \omega^2} \cos \omega L \sin \Omega L
\]

\[\Rightarrow \sin \Omega L = 0\]

which reproduces (305).
A hydrodynamic stability example

- We shall consider the weakly nonlinear theory for plane Poiseuille flow, $U(y) = 1 - y^2$, between plates at $y = \pm 1$.
- We shall consider two-dimensional disturbances, and hence use a streamfunction, $\psi$, where

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (312)$$

- The Navier-Stokes equations are now

$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \nabla^2 \psi - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi. \quad (313)$$
Finding the form of the expansion

• We shall consider waves that grow or decay **slowly** compared with their period of oscillation.

• To a first approximation the frequency is then real.

• This occurs near a neutral curve.

• We can determine the form of the perturbation expansion by first adding a small neutral wave to the basic flow:

\[
\psi = \int U \, dy + \epsilon \psi_{11}(y)E + \epsilon \bar{\psi}_{11}(y)E^{-1}
\]  

(314)

where \( \bar{\psi}_{11} \) is the complex conjugate of \( \psi \), \( E = \exp(i(\alpha x - \omega t)) \), \( \alpha \) and \( \omega \) are real, and \( \epsilon \) characterises the **small amplitude** of the disturbance.

• Adding the complex conjugate of the wavy term ensures that \( \psi \) is real.
Substitute (314) into (313) and equate coefficients of powers of $\epsilon$ and $E$:

At $\epsilon E$ the Orr-Sommerfeld equation is obtained:

$$(U - c)(\psi''_{11} - \alpha^2 \psi_{11}) - U'' \psi_{11} - \frac{1}{i\alpha \text{Re}}(\psi'''_{11} - 2\alpha^2 \psi''_{11} + \alpha^4 \psi_{11}) = 0$$

(315)

where $c = \omega/\alpha$, (at this order we reproduce linear theory).

At $\epsilon^2 E^0$:

$$(\bar{\psi}_{11} \psi'_{11} - \psi_{11} \bar{\psi}'_{11})'' = 0.$$  

(316)

At $\epsilon^2 E^2$:

$$\psi'_{11} \psi''_{11} - \psi_{11} \psi'''_{11} = 0.$$  

(317)

$\psi_{11}$ is determined by solving (315), leaving (316) and (317) unsatisfied.
• (316) and (317) can be solved by introducing new terms proportional to $\epsilon^2 E^0$ and $\epsilon^2 E^2$ to the expansion (314).

• These new terms produce terms of $O(\epsilon^3)$, so terms proportional to $\epsilon^3 E$ and $\epsilon^3 E^3$ must also be included:

$$\psi = \int U \, dy + \epsilon \psi_{11}(y)E + \epsilon^2 \psi_{02} + \epsilon^2 \psi_{22}E^2 + \epsilon^3 \psi_{13}E$$

$$+ \epsilon^3 \bar{\psi}_{33}E^3 + \epsilon^2 \bar{\psi}_{11}(y)E^{-1} + \epsilon^2 \bar{\psi}_{22}E^{-2} + \epsilon^3 \bar{\psi}_{13}E^{-1}$$

$$+ \epsilon^3 \bar{\psi}_{33}E^{-3}$$  \hspace{1cm} (318)

where complex conjugates have been added to make $\psi$ real.

• Substitute (318) into (313) and equate powers of $\epsilon$ and $E$:
• At $\epsilon E$:

$$L_1(\psi_{11}) = 0 \quad (319)$$

where we introduce $L_n$ to denote the Orr-Sommerfeld operator for wavenumber and frequency $(n\alpha, n\omega)$, i.e. (319) is the same as (315).

• At $\epsilon^2 E^0$:

$$\frac{\psi_{02}^{\\''''}}{i\alpha Re} = (\bar{\psi}_{11}\psi_{11}' - \psi_{11}\bar{\psi}_{11}')'' \quad (320)$$

• At $\epsilon^2 E^2$:

$$L_2(\psi_{22}) = \psi_{11}' \psi_{11}'' - \psi_{11} \psi_{11}''' \quad (321)$$
• At $\epsilon^3 E$:

\[
L_1(\psi_{13}) = \psi''_{11} \psi'_{02} - \psi_{11}(\alpha^2 \psi'_{02} + \psi'''_{02}) + \bar{\psi}_{11} \psi''_{22} - \bar{\psi}''_{11} \psi'_{22} + 2\bar{\psi}'_{11} \psi''_{22} - 2(3\alpha^2 \bar{\psi}'_{11} + \bar{\psi}'''_{11})\psi_{22} - 3\alpha^2 \bar{\psi}_{11} \psi'_{22} \tag{322}
\]

• At $\epsilon^3 E^3$:

\[
L_3(\psi_{33}) = \ldots \tag{323}
\]

• As before, (319) is solved for $\psi_{11}$.

• (320) is then solved for the mean flow correction term $\psi_{02}$. 
• (321) is solved for $\psi_{22}$, because, in general, if $(\alpha, \omega)$ are eigenvalues of the OS equation, then $(2\alpha, 2\omega)$ will not be eigenvalues.

• Similarly, (323) can be solved for $\psi_{33}$, because, in general, if $(\alpha, \omega)$ are eigenvalues of the OS equation, then $(3\alpha, 3\omega)$ will not be eigenvalues.

• However, an important difficulty arises when we consider (322).

• $(\alpha, \omega)$ are eigenvalues of the left hand side of (322), so there is no solution, unless the right hand side satisfies a solvability condition similar to (311).

• In general, the solvability condition for (322) will not be satisfied.

• Therefore, our perturbation expansion needs to be modified...
Multiple scales theory

- It was shown by Stuart (1960, JFM) that this difficulty can be resolved if one allows the amplitude of the fundamental wave to vary on an appropriate slow time scale.
- In fact, this assumption is implicit in our earlier figures showing nonlinear saturation and threshold behaviour: the envelope of the oscillation varies slowly compared with the time scale of the oscillation.
- The fundamental wave is written

\[ \epsilon A(T)\psi_{11}(y)E \]  \hspace{1cm} (324)

where the slow time scale for the amplitude evolution is

\[ T = \epsilon^2 t. \]  \hspace{1cm} (325)
• Furthermore, we need no longer consider exactly neutral waves: we can let the eigenvalue for a given real $\alpha$ be

$$\omega = \omega_r + i\epsilon^2 \omega_i$$  \hspace{1cm} (326)

(e.g. by being appropriately close to the neutral curve).

• The term that depends on the fast time scale is now

$$E = \exp i(\alpha x - \omega_r t).$$ \hspace{1cm} (327)

• Differentiating (324) with respect to $t$ gives

$$\frac{\partial}{\partial t} \left[ \epsilon A(T) \psi_{11}(y) E \right] = \epsilon \frac{dA}{dT} \frac{dT}{dt} \psi_{11} E - i \omega_r \epsilon A \psi_{11} E$$

$$= -i \omega \epsilon A \psi_{11} E + \epsilon^3 \left( \frac{dA}{dT} - \omega_i A \right) \psi_{11} E.$$ \hspace{1cm} (328)
• The slow scale, (325), and the smallness of the imaginary part of frequency, (326), were chosen to produce terms proportional to $\epsilon^3 E$ in (328).

• These terms would therefore appear in (322), where a solvability condition is needed.

• We choose $A(T)$ to vary so as to satisfy the solvability condition.

• Therefore, we replace (318) by

$$
\psi = \int U \, dy + \epsilon A(T) \psi_{11}(y) E + \epsilon^2 |A|^2 \psi_{02} + \epsilon^2 A^2 \psi_{22} E^2 \\
+ \epsilon^3 \psi_{13} E + \epsilon^3 \psi_{33} E^3 + \epsilon \bar{A} \bar{\psi}_{11} E^{-1} + \epsilon^2 \bar{A}^2 \bar{\psi}_{22} E^{-2} \\
+ \epsilon^3 \bar{\psi}_{13} E^{-1} + \epsilon^3 \bar{\psi}_{33} E^{-3}.
$$

(329)
Substitute (329) into the Navier-Stokes equations (313) and at $\varepsilon^3 E$ we get:

$$L_1(\psi_{13}) = \left[ \left( 3i\alpha U + i\frac{U''}{\alpha} - 2i\omega + \frac{4\alpha^2}{Re} \right) \psi_{11} 
- \left( i\frac{U}{\alpha} + \frac{4}{Re} \right) \psi_{11}'' \right] \left( \frac{dA}{dT} - \omega_i A \right) 
+ [\psi_{11}'\psi_{02}' - \psi_{11}(\alpha^2\psi_{02}' + \psi_{02}'') + \bar{\psi}_{11}\psi_{22}'''
- \bar{\psi}_{11}'\psi_{22}'' + 2\bar{\psi}_{11}'\psi_{22}'' - 2(3\alpha^2\bar{\psi}_{11}' + \bar{\psi}_{11}'')\psi_{22} 
- 3\alpha^2\bar{\psi}_{11}'\psi_{22}'] A|A|^2 
= f(y) \left( \frac{dA}{dT} - \omega_i A \right) + h(y)A|A|^2. \quad (330)$$
An amplitude equation

- We apply the solvability condition (311), i.e. multiply (330) by the adjoint of the Orr-Sommerfeld equation, $g(y)$, and integrate across the flow:

$$0 = \int_{-1}^{1} g(y)f(y)dy \left( \frac{dA}{dT} - \omega_i A \right) + \int_{-1}^{1} g(y)h(y)dy A|A|^2$$

(331)

which can be rearranged to give

$$\frac{dA}{dT} = \omega_i A + \lambda A|A|^2$$

(332)

where

$$\lambda = -\frac{\int_{-1}^{1} g(y)h(y)dy}{\int_{-1}^{1} g(y)f(y)dy}.$$  

(333)
• The form of (332) was conjectured by Landau in 1944, but it was not derived until Stuart in 1960.

• (332) is called the Landau equation, or Stuart-Landau equation.

• It is an example of an amplitude equation.

• $\lambda$ is called the Landau coefficient.

• The sign of $\Re(\lambda)$ determines whether nonlinearity is stabilizing or destabilizing.

• The adjoint eigenfunction, $g(y)$, is found by a similar procedure as that leading to (310): multiply (315) by $g$, integrate across the flow and use integration by parts to transfer derivatives from $\psi_{11}$ to $g$. 
• The adjoint of the Orr-Sommerfeld equation is found to be

\[(U - c)(g'' - \alpha^2 g) + 2U'g' - \frac{1}{i\alpha\text{Re}}(g'''' - 2\alpha^2 g'' + \alpha^4 g) = 0,\]  
(334)

cf. the adjoint of the Rayleigh equation (48).

• We can analyse the Landau equation (332) by considering the magnitude and phase of \(A\).

• Substitute \(A = r \exp i\theta\) into (332):

\[
\frac{dA}{dT} = \frac{dr}{dT} \exp i\theta + i \frac{d\theta}{dT} r \exp i\theta = \omega_i r \exp i\theta + (\lambda_r + i\lambda_i) r^3 \exp i\theta
\]  
(335)

where \(\lambda_r\) and \(\lambda_i\) are the real and imaginary parts of \(\lambda\).
• Dividing (335) by \( \exp i \theta \) then equating real and imaginary parts gives

\[
\frac{dr}{dT} = \omega_i r + \lambda_r r^3 \quad \text{(336)}
\]
\[
\frac{d\theta}{dT} = \lambda_i r^2. \quad \text{(337)}
\]

• The condition for an equilibrium amplitude is

\[
\frac{dr}{dT} = 0 \quad \Rightarrow \quad r = 0, \quad r = \left(\frac{-\omega_i}{\lambda_r}\right)^{1/2} \quad \text{(338)}
\]

where the nontrivial solution exists when \( \omega_i / \lambda_r < 0 \).

• The stability of these equilibria can be found by considering the sign of \( dr/dT \) for various \( r \).
• Solid lines are stable; dashed lines are unstable.
• For small $r$, $\frac{dr}{dT} \sim \omega_i r$, so the $r = 0$ solution follows linear theory, i.e. unstable if $\omega_i > 0$ and stable if $\omega_i < 0$.
• At large $r$, $\frac{dr}{dT} \sim \lambda_r r^3$, so the nonlinear solution is unstable if $\lambda_r > 0$ (threshold behaviour) and stable if $\lambda_r < 0$ (nonlinear saturation).
Wave interaction

• Nonlinearity does not only modify the growth rate of a single wave, e.g. to create nonlinear equilibrium solutions.
• If more than one wave is present, then nonlinearity causes them to interact with each other (the superposition principle does not apply to nonlinear equations).
• Consider a pair of waves with wavy parts

\[ E_1 = \exp \left( i \alpha_1 x - \omega_1 t \right), \quad E_2 = \exp \left( i \alpha_2 x - \omega_2 t \right) \]  

(339)

where \((\alpha_1, \omega_1)\) are eigenvalues, and so are \((\alpha_2, \omega_2)\), that do not satisfy resonance conditions, e.g. \(\omega_2 \neq n\omega_1\) when \(\alpha_2 = n\alpha_1\) for some integer \(n\).
• By following a similar procedure to that for a single wave, we find that an expansion of the form

\[ \psi = \int U \, dy + \epsilon A_1(T) \psi_{111}(y) E_1 + \epsilon A_2(T) \psi_{211}(y) E_2 + \ldots \]  

(340)
...leads to two solvability conditions, giving a pair of coupled amplitude equations:

\[
\begin{align*}
\frac{dA_1}{dT} &= \omega_1 A_1 + \lambda_1 |A_1|^2 A_1 + a_1 |A_2|^2 A_1 & (341) \\
\frac{dA_2}{dT} &= \omega_2 A_2 + \lambda_2 |A_2|^2 A_2 + a_2 |A_1|^2 A_2 & (342)
\end{align*}
\]

The Landau coefficients \(\lambda_1\) and \(\lambda_2\) govern the evolution of \(A_1\) and \(A_2\) if each is present by itself, i.e. if \(A_1 = 0\) or \(A_2 = 0\).

By letting \(A_1 = r_1 \exp i\theta_1\) and \(A_2 = r_2 \exp i\theta_2\) we find that the real parts of the interaction coefficients \(a_1\) and \(a_2\) determine how energy is transferred from one wave to the other.
Resonant wave interaction

- If the dispersion relation is such that it admits roots satisfying

\[ \alpha_2 = 2\alpha_1 \quad \text{and} \quad \omega_2 = 2\omega_1 \quad (343) \]

or

\[ \alpha_2 = 3\alpha_1 \quad \text{and} \quad \omega_2 = 3\omega_1 \quad (344) \]

then the form of the amplitude equations changes qualitatively.

- We shall consider the case (343).

- It turns out that solvability conditions now first occur at \( O(\epsilon^2) \), instead of the \( O(\epsilon^3) \) that we encountered before.
• This means that the slow time scale is now $T = \epsilon t$, and the imaginary parts of $\omega_1$ and $\omega_2$ are $O(\epsilon)$.

• The solvability conditions give amplitude equations

\[
\frac{dA_1}{dT} = \omega_1 i A_1 + b_1 A_2 \bar{A}_1 \tag{345}
\]

\[
\frac{dA_2}{dT} = \omega_2 i A_2 + b_2 A_1^2. \tag{346}
\]

• The direction of energy transfer between the waves now depends on the relative phase between the waves.

• A detuning parameter can be introduced to account for the case when the real parts of the eigenvalues don’t satisfy (343) exactly, but differ by an amount $O(\epsilon)$.

• This adds an imaginary coefficient to either $A_1$ or $A_2$. 
Craik resonant triads

- Craik (1971, JFM) showed that a 2 : 1 resonance like (343) can be satisfied in typical boundary layer dispersion relations, for any given plane wave and a particular pair of oblique waves with suitably chosen spanwise wavenumber:

\[ A_1 \exp i(\alpha x - \omega t), \quad A_2 \exp i\left(\frac{\alpha}{2}x + \beta z - \frac{\omega}{2} t\right), \quad A_3 \exp i\left(\frac{\alpha}{2}x - \beta z - \frac{\omega}{2} t\right). \]

This is often called a subharmonic resonance.

- The resonant amplitude equations are in the form

\[ \dot{A}_1 = a_1 A_1 + b_1 A_2 A_3 \quad \text{(348)} \]
\[ \dot{A}_2 = a_2 A_2 + b_2 A_1 \bar{A}_3 \quad \text{(349)} \]
\[ \dot{A}_3 = a_3 A_3 + b_3 A_1 \bar{A}_2. \quad \text{(350)} \]
• For example, long waves near the upper branch of the neutral curve have leading order inviscid dispersion relation given by \( c = \alpha / U'_0 \), see (155), and the generalization to 3d waves proportional to \( \exp i(\alpha x + \beta z - \omega t) \) is

\[
c = \sqrt{\frac{\alpha^2 + \beta^2}{U'_0}}
\]

where \( c = \omega / \alpha \).

• The phase velocities of the waves in (347) are all equal.

• Therefore, for a 2d wave with wavenumber \( \alpha_0 \), the 3d waves satisfying (347) and (351) are given by

\[
c = \frac{\alpha_0}{U'_0} = \sqrt{\frac{(\alpha_0/2)^2 + \beta_0^2}{U'_0}} \quad \Rightarrow \quad \beta_0 = \pm \frac{\sqrt{3}}{2} \alpha_0.
\]

• The wave-angle of these resonant oblique waves is \( \theta = \pi / 3 \), where \( \tan \theta = \beta / \alpha = (\sqrt{3} \alpha_0 / 2) / (\alpha_0 / 2) = \sqrt{3} \).

• This wave-angle prediction is independent of \( \alpha_0 \) and \( U'_0 \), and can be tested in experiments.
• Resonant waves develop nonlinear behaviour more quickly than non-resonant waves because they evolve on the faster (less slow) time scale $T = \epsilon t$, compared to the non-resonant time scale $T = \epsilon^2 t$.

• Therefore, from broadband initial background disturbances, those waves that experience resonant nonlinear growth may quickly dominate the dynamics.

• Resonant interaction can be detected in experiments by checking for phase dependence, e.g. by changing the sign of the input disturbance: (341) and (342) are invariant under $A_1 \rightarrow -A_1$ and $A_2 \rightarrow -A_2$, but (345) and (346), and (348) – (350), behave differently under a change of sign.

• Experiments have confirmed that Craik resonant triads are important in generating three-dimensionality in the early stages of the breakdown to turbulence.

• But predictions for the wave-angle are not very accurate.