2. Inviscid stability theory

- The Rayleigh equation:

\[(U - c)(v'' - \alpha^2 v) - U''v = 0.\]  \[(38)\]

- Rayleigh's inflexion point theorem.

- Divide (38) by \((U - c)\), multiply by \(\bar{v}\) (the complex conjugate of \(v\)) and integrate across the flow domain:

\[\int_{y_1}^{y_2} \bar{v}v'' - \left(\frac{U''}{U - c} + \alpha^2\right)|v|^2 \, dy = 0.\]  \[(39)\]

- Integrate the first term by parts:

\[\left[\bar{v}v'\right]_{y_1}^{y_2} + \int_{y_1}^{y_2} -\bar{v}'v' - \left(\frac{U''}{U - c} + \alpha^2\right)|v|^2 \, dy = 0\]  \[(40)\]

\[\Rightarrow \int_{y_1}^{y_2} |v'|^2 + \alpha^2|v|^2 \, dy + \int_{y_1}^{y_2} \frac{U''|v|^2}{U - c} \, dy = 0.\]  \[(41)\]
Consider the imaginary part of (41), taking $\alpha$ real, with $c$ possibly complex:

$$c_i \int_{y_1}^{y_2} \frac{U'' |v|^2}{|U - c|^2} \, dy = 0. \quad (42)$$

Therefore, for instability, i.e. $c_i \neq 0$, the integral in (42) must be zero, which can only be true if either $U'' \equiv 0$, or $U''$ changes sign at least once in the interval $y_1 < y < y_2$.

This proves Rayleigh’s (1880) inflexion point theorem:

A necessary, but not sufficient, condition for instability is that the velocity profile have an inflexion point.
Fjørtoft’s theorem

• Let there be an inflexion point at $y = y_I$, and let $U_I = U(y_I)$.
• Note that if $c_i \neq 0$, then (42) implies

$$ (c_r - U_I) \int_{y_1}^{y_2} \frac{U''|v|^2}{|U - c|^2} \, dy = 0. \quad (43) $$

• The real part of (41) is

$$ \int_{y_1}^{y_2} \frac{U''(U - c_r)|v|^2}{|U - c|^2} \, dy = - \int_{y_1}^{y_2} |v'|^2 + \alpha^2 |v|^2 \, dy. \quad (44) $$

• Adding (43) and (44) leads to

$$ \int_{y_1}^{y_2} \frac{U''(U - U_I)|v|^2}{|U - c|^2} \, dy < 0. \quad (45) $$
Equation (45) proves Fjørtoft’s (1950) theorem:

A necessary, but not sufficient, condition for instability is that $U''(U - U_I) < 0$ somewhere in the flow.

Examples of Rayleigh’s and Fjørtoft’s results:

- Stable by Rayleigh
  - $U'' < 0$
  - $U'' > 0$

- Stable by Fjørtoft
  - $U'' < 0$
  - $U'' > 0$
  - $U''(U - U_I) > 0$

- Could be unstable
  - $U'' > 0$
  - $U''(U - U_I) < 0$
Howard’s semi-circle theorem

• This is a result placing bounds on the phase velocity and the maximum growth rate of any unstable wave.

• Howard’s semi-circle theorem is derived in a similar manner to Rayleigh’s and Fjørtoft’s, but from the adjoint of the Rayleigh equation.

• The adjoint equation has the same eigenvalues as the Rayleigh equation.

• First derive the adjoint of the Rayleigh equation:

• Multiply (38) by a function \( w \) satisfying the same homogeneous boundary conditions as \( v \) and integrate:

\[
\int_{y_1}^{y_2} \left[ (U - c)(v'' - \alpha^2 v) - U'' v \right] w \, dy = 0. \tag{46}
\]
• Integrating the \( v'' \) term in (46) by parts twice and applying homogeneous boundary conditions gives

\[
\int_{y_1}^{y_2} \left[ (U - c)(w'' - \alpha^2 w) + 2U'w' \right] v \, dy = 0. \tag{47}
\]

• Let

\[
(U - c)(w'' - \alpha^2 w) + 2U'w' = 0. \tag{48}
\]

• (48) is the adjoint of the Rayleigh equation (38).

• It may be verified that if \( v \) is a solution of (38), then \( w = v / (U - c) \) is a solution of (48).

• The eigenvalues of (48) are the same as the eigenvalues of (38) because \( w \) satisfies the same boundary conditions as \( v \).

• (A bi-orthogonality condition exists between the eigenfunctions of an equation and its adjoint, which allows coefficients to be calculated in eigenfunction expansions, which is useful in solving initial value problems).
• Multiply the adjoint Rayleigh equation (48) by \((U - c)\) and write the result as

\[
[(U - c)^2 w']' - \alpha^2 (U - c)^2 w = 0. \tag{49}
\]

• Multiply (49) by \(\bar{w}\) and integrate:

\[
\int_{y_1}^{y_2} \bar{w} [(U - c)^2 w']' - \alpha^2 (U - c)^2 |w|^2 \, dy = 0. \tag{50}
\]

• Integrating the first term once by parts gives

\[
\int_{y_1}^{y_2} (U - c)^2 Q \, dy = 0 \tag{51}
\]

where \(Q = |w'|^2 + \alpha^2 |w|^2\) is positive definite.
• Equating the imaginary parts of (51) gives

\[
\int_{y_1}^{y_2} UQ \, dy = c_r \int_{y_1}^{y_2} Q \, dy. \tag{52}
\]

• Equating the real parts of (51), and making use of (52), gives

\[
\int_{y_1}^{y_2} U^2 Q \, dy = (c_r^2 + c_i^2) \int_{y_1}^{y_2} Q \, dy. \tag{53}
\]

• Let \( U_{\text{min}} \) and \( U_{\text{max}} \) be the minimum and maximum velocities respectively of the basic flow, then clearly

\[
\int_{y_1}^{y_2} (U - U_{\text{min}})(U - U_{\text{max}})Q \, dy \leq 0. \tag{54}
\]
Multiplying out the brackets in (54) and substituting in (52) and (53) gives

\[
\left[ c_r^2 + c_i^2 - (U_{\text{min}} + U_{\text{max}})c_r + U_{\text{min}}U_{\text{max}} \right] \int_{y_1}^{y_2} Q \, dy \leq 0. \tag{55}
\]

Therefore

\[
\left[ c_r^2 + c_i^2 - (U_{\text{min}} + U_{\text{max}})c_r + U_{\text{min}}U_{\text{max}} \right] \leq 0, \tag{56}
\]

which can be arranged to give

\[
\left[ c_r - \frac{1}{2}(U_{\text{min}} + U_{\text{max}}) \right]^2 + c_i^2 \leq \left[ \frac{1}{2}(U_{\text{max}} - U_{\text{min}}) \right]^2. \tag{57}
\]
Equation (57) proves Howard’s (1961) semi-circle theorem for unstable waves:

The complex phase velocity $c$ lies inside, or on, the semi-circle centred on $(U_{\text{max}} + U_{\text{min}})/2$ with radius $(U_{\text{max}} - U_{\text{min}})/2$: 

![Diagram showing a semi-circle with a dot inside or on it, representing the complex phase velocity $c$.]
In general, the difficulty in solving the Rayleigh equation (38) lies in the $y$-dependence of the coefficients. However, for profiles made up of linear segments, $U'' = 0$, and then (38) reduces to

$$v'' - \alpha^2 v = 0$$

(58)

This suggests a method for estimating the stability of smooth profiles by approximating them by piecewise-linear profiles, e.g.
Jump conditions

- Solutions to (58) in neighbouring segments of the basic profile must be related to one another.
- Let either $U$ or $U'$ be discontinuous at $y = y_0$.
- Let
  $$ \Delta f = \lim_{\epsilon \to 0} \{ f(y_0 + \epsilon) - f(y_0 - \epsilon) \} $$
  (59)
  be the jump in a quantity $f$ across $y_0$.
- Note that the Rayleigh equation (38) can be written
  $$ [(U - c)v' - U'v]' - \alpha^2 (U - c)v = 0. $$
  (60)
• Integrate (60) across the discontinuity from \( y_0 - \epsilon \) to \( y_0 + \epsilon \):

\[
[(U - c)v' - U'v]_{y_0-\epsilon}^{y_0+\epsilon} - \alpha^2 \int_{y_0-\epsilon}^{y_0+\epsilon} (U - c)v \, dy = 0. \tag{61}
\]

• In the limit \( \epsilon \to 0 \) the integral term in (61) vanishes, giving the first jump condition:

\[
\Delta [(U - c)v' - U'v] = 0. \tag{62}
\]

• Note that eliminating \( u \) between (28) and (29) (with \( 1/Re \to 0 \)) gives

\[
i\alpha p = (U - c)v' - U'v. \tag{63}
\]

• Therefore (62) corresponds to continuous pressure across the discontinuity (dynamic boundary condition).
• Divide (63) by \((U - c)^2\):

\[
\frac{i\alpha p}{(U - c)^2} = \frac{v'}{U - c} - \frac{U'v}{(U - c)^2} = \left[ \frac{v}{U - c} \right]'.
\] (64)

• Integrating (64) across the discontinuity from \(y_0 - \epsilon\) to \(y_0 + \epsilon\), taking the limit \(\epsilon \to 0\) and noting that the integral on the LHS vanishes, gives the second jump condition

\[
\Delta \left[ \frac{v}{U - c} \right] = 0.
\] (65)

• When \(c\) is real, \(v/(U - c)\) is the ratio of vertical to horizontal velocity in a frame of reference moving at \(c\), and so gives the slope of the streamlines in this frame.

• Therefore, (61) implies that the streamlines on one side of the discontinuity are parallel to those on the other (corresponds to a kinematic boundary condition).
Kelvin-Helmholtz instability

- Our first example is the simplest model of a mixing layer:

\[ U = \begin{cases} U_1 & \text{if } y > 0 \\ U_2 & \text{if } y < 0 \end{cases} \]

- In each layer the Rayleigh eqn (38) reduces to \( v'' - \alpha^2 v = 0 \).

- Let

\[ v = \begin{cases} v_1 & \text{if } y > 0 \\ v_2 & \text{if } y < 0 \end{cases} \] \hspace{1cm} (66)

- The solutions satisfying homogeneous boundary conditions \( v_1 \to 0 \) as \( y \to \infty \), and \( v_2 \to 0 \) as \( y \to -\infty \), are

\[ v_1 = Ae^{-\alpha y}, \quad v_2 = Be^{\alpha y} \] \hspace{1cm} (67)

where \( \alpha > 0 \).
• Applying the first jump condition (62) to the solutions (67) gives

\[(U_1 - c)v_1'(0) = (U_2 - c)v_2'(0) \quad (68)\]
\[\Rightarrow (U_1 - c)(-\alpha)A = (U_2 - c)\alpha B \quad (69)\]

• Applying the second jump condition (65) to (67) gives

\[
\frac{v_1(0)}{(U_1 - c)} = \frac{v_2(0)}{(U_2 - c)} \quad (70)
\]
\[\Rightarrow \frac{A}{(U_1 - c)} = \frac{B}{(U_2 - c)} \quad (71)\]
• Eliminating $A/B$ between (69) and (71) leads to the dispersion relation for Kelvin-Helmholtz instability on a vortex sheet:

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{i}{2}(U_1 - U_2).$$

(72)

• There is instability for all $U_1 \neq U_2$.

• The phase velocity is the mean of $U_1$ and $U_2$.

• In fact, $c$ lies at the top of Howard’s semi-circle.

• The flow is nondispersive: $c$ is independent of $\alpha$.

• The growth rate $\omega_i = \alpha c_i = \alpha |U_1 - U_2|$, i.e. the shorter the wave the greater the growth rate.

• This nonphysical growth arises because there is no natural length scale in the basic flow.
Mixing layer with finite thickness

- Consider a mixing layer of thickness $h$:

$$U = \begin{cases} 
U_1 & \text{if } y > h/2 \\
(U_1 + U_2)/2 + (U_1 - U_2)y/h & \text{if } -h/2 < y < h/2 \\
U_2 & \text{if } y < -h/2
\end{cases} \quad (73)$$

- Let

$$v = \begin{cases} 
v_1 & \text{if } y > h/2 \\
v_2 & \text{if } -h/2 < y < h/2 \\
v_3 & \text{if } y < -h/2
\end{cases} \quad (74)$$
• In each layer the Rayleigh eqn (38) reduces to $v'' - \alpha^2 v = 0$.

• The solutions satisfying homogeneous boundary conditions are

$$v_1 = A e^{-\alpha y} \quad (75)$$
$$v_2 = B e^{-\alpha y} + C e^{\alpha y} \quad (76)$$
$$v_3 = D e^{\alpha y} \quad (77)$$

• Applying the first jump condition (62) to solutions (75) and (76) at $y = h/2$ gives

$$-(U_1 - c)\alpha A e^{-\alpha h/2} = (U_1 - c) \left( -\alpha B e^{-\alpha h/2} + \alpha C e^{\alpha h/2} \right) - \frac{U_1 - U_2}{h} \left( B e^{-\alpha h/2} + C e^{\alpha h/2} \right). \quad (78)$$
• Applying the second jump condition (65) to solutions (75) and (76) at \( y = \frac{h}{2} \) gives

\[
\frac{A e^{-\alpha h/2}}{U_1 - c} = \frac{B e^{-\alpha h/2} + C e^{\alpha h/2}}{U_1 - c}.
\]  

(79)

• Applying jump conditions (62) and (65) to solutions (76) and (77) at \( y = -\frac{h}{2} \) gives

\[
(U_2 - c) \alpha D e^{-\alpha h/2} = (U_2 - c) \left( -\alpha B e^{\alpha h/2} + \alpha C e^{-\alpha h/2} \right) - \frac{U_1 - U_2}{h} \left( B e^{\alpha h/2} + C e^{-\alpha h/2} \right)
\]  

(80)

\[
\frac{D e^{-\alpha h/2}}{U_2 - c} = \frac{B e^{\alpha h/2} + C e^{-\alpha h/2}}{U_2 - c}.
\]  

(81)
Eliminating $A$, $B$, $C$ and $D$ from (78), (79), (80) and (81) gives the dispersion relation

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{U_1 - U_2}{2\alpha h} \left[(1 - \alpha h)^2 - e^{-2\alpha h}\right]^{1/2}. \quad (82)$$

In the long-wave limit, i.e. $\alpha h \ll 1$, (82) reduces to

$$c \sim \frac{1}{2}(U_1 + U_2) \pm \frac{i}{2}(U_1 - U_2). \quad (83)$$

Therefore, for waves that are long compared with the shear layer thickness there is instability, and the Kelvin-Helmholtz vortex sheet is a good model.
• In the short-wave limit, i.e. \( \alpha h \gg 1 \), (82) reduces to

\[
c \sim U_1 - \frac{U_1 - U_2}{2\alpha h}, \quad U_2 + \frac{U_1 - U_2}{2\alpha h}.
\]

\( (84) \)

• Therefore waves that are short compared with the shear layer thickness are stable and have phase velocities close to the free stream velocities (the joins in the profile act as waveguides).

• The change from stable to unstable behaviour takes place for wavelengths that are comparable to the shear layer thickness, i.e. \( \alpha h = O(1) \).

• The neutral wavenumber is \( \alpha h = 1.278 \).

• The most unstable wave is on the same scaling: its wavenumber is \( \alpha h = 0.7968 \) and the strongest growth rate is \( 0.2012(U_1 - U_2)/h \).
Model boundary layer

• Consider a model boundary layer of thickness $\delta$:

\[
U = \begin{cases} 
U_1 & \text{if } y > \delta \\
\frac{U_1 y}{\delta} & \text{if } 0 < y < \delta
\end{cases}
\]

(85)

• Let

\[
v = \begin{cases} 
v_1 & \text{if } y > \delta \\
v_2 & \text{if } 0 < y < \delta
\end{cases}
\]

(86)

• Solutions to the reduced Rayleigh equation (38),

\[v'' - \alpha^2 v = 0,\] satisfying homogeneous boundary conditions:

\[
v_1 = Ae^{-\alpha y}
\]

(87)

\[
v_2 = B \left( e^{-\alpha y} - e^{\alpha y} \right)
\]

(88)
Applying the jump conditions (62) and (65) to the solutions (87) and (88) at \( y = \delta \) gives

\[
-(U_1 - c)\alpha A e^{-\alpha \delta} = -(U_1 - c)\alpha B \left( e^{-\alpha \delta} + e^{\alpha \delta} \right)
- \frac{U_1}{\delta} B \left( e^{-\alpha \delta} - e^{\alpha \delta} \right)
\]

\[
\frac{A e^{-\alpha \delta}}{U_1 - c} = \frac{B \left( e^{-\alpha \delta} - e^{\alpha \delta} \right)}{U_1 - c}
\]

Eliminating \( A \) and \( B \) from (89) and (90) gives the dispersion relation

\[
c = \frac{U_1}{2\alpha \delta} \left( e^{-2\alpha \delta} - 1 + 2\alpha \delta \right),
\]

showing this boundary layer is stable for all \( \alpha \).

In the long-wave limit, i.e. \( \alpha \delta \ll 1 \), we have

\[
c \sim \alpha \delta U_1.
\]
Tollmien’s critical point solutions

• In fact, basic flow curvature, $U''$, can make an important contribution to the (in)stability of smooth velocity profiles.

• Tollmien (1929) showed that this contribution comes from critical points $y_c$ defined by $U(y_c) = c$.

• Critical points are regular singular points of the Rayleigh equation (38).

• Seek series solutions about $y_c$: assume

$$v = \sum_{n=0} a_n(y - y_c)^n. \quad (93)$$

• Expand the basic flow about $y_c$:

$$U = c + (y - y_c)U' + \frac{1}{2}(y - y_c)^2U'' + \ldots \quad (94)$$

where $U'_c = U'(y_c) \neq 0$, $U''_c = U''(y_c)$ etc.
• Substituting expansions (93) and (94) into the Rayleigh equation (38), and equating coefficients of powers of \( (y - y_c) \) gives

\[
a_0 = 0, \quad a_2 = \frac{U''}{2U'} a_1, \quad a_3 = \frac{1}{6} \left( \frac{U'''}{U'} + \alpha^2 \right) a_1, \ldots \quad (95)
\]

• The constant \( a_1 \) is arbitrary since (38) is a linear homogeneous equation.

• We can choose to normalize this solution such that \( a_1 = 1 \), and denote it by

\[
v_1 = (y - y_c) + \frac{U''}{2U'} (y - y_c)^2 + \frac{1}{6} \left( \frac{U'''}{U'} + \alpha^2 \right) (y - y_c)^3 + \ldots \quad (96)
\]
To find a second independent solution to the Rayleigh equation (38), consider $a_0 \neq 0$.

The leading terms in (38) near $y = y_c$ are now

$$U'_c(y - y_c) (v'' - \alpha^2 a_0) - U''_c a_0 = 0. \quad (97)$$

Balance at leading order now requires

$$v'' \sim (y - y_c)^{-1} \Rightarrow v \sim (y - y_c) \ln(y - y_c). \quad (98)$$

Therefore, let

$$v = 1 + b_1 v_1 \ln(y - y_c) + \sum_{n=2} b_n (y - y_c)^n \quad (99)$$

where we have chosen the normalization $a_0 = 1$, there is no coefficient of $O(y - y_c)$ since this is contained in the $v_1$ solution, and the use of $v_1$ in the coefficient of $\ln(y - y_c)$ is acceptable since $v_1 \sim (y - y_c)$. 
• Substitute (99) into (38): the coefficient of \( \ln(y - y_c) \) vanishes since \( v_1 \) satisfies (38).

• Equating coefficients of powers of \( (y - y_c) \) gives the \( b_n \). Calling this solution \( v_2 \), we find

\[
\begin{align*}
v_2 &= 1 + \frac{U_c''}{U_c'} v_1 \ln(y - y_c) \\
&\quad + \frac{1}{2} \left[ \frac{U_c'''}{U_c'} - \left( \frac{2U_c''}{U_c'} \right)^2 + \alpha^2 \right] (y - y_c)^2 + \ldots (100)
\end{align*}
\]

• \( v_1 \), (96), is the regular solution and \( v_2 \), (100), is the singular solution.

• The general solution to the Rayleigh equation (38) near a critical point \( y_c \) is

\[
v = Av_1 + Bv_2 \quad \text{(101)}
\]

where \( A \) and \( B \) are arbitrary constants.
Path around a critical point

- The presence of the logarithm raises the question of which branch to use when \( y < y_c \) when \( y_c \) is real.
- I.e. should we take \( \ln(y - y_c) = \ln(y_c - y) + i\pi \) or \( \ln(y - y_c) = \ln(y_c - y) - i\pi \)?
- This question was first answered by looking at the viscous version of the problem, i.e. the Orr-Sommerfeld equation.
- Viscosity removes the singularity in the solution. Its effects were originally understood using WKB theory, though matched asymptotic expansions can also be used. The branch is chosen that corresponds to the viscous solution.
- This question can also be answered without appeal to viscosity, but by considering the inviscid initial value problem.
• When $\alpha > 0$, it turns out (Lin’s critical point rule) that

\[ U'_c > 0 \implies \text{solution path passes below critical point} \]
\[ \implies \ln(y - y_c) = \ln(y_c - y) - i\pi \quad \text{for } y < y_c \quad (102) \]

• and

\[ U'_c < 0 \implies \text{solution path passes above critical point} \]
\[ \implies \ln(y - y_c) = \ln(y_c - y) + i\pi \quad \text{for } y < y_c. \quad (103) \]
The imaginary term generated by the logarithm, $\pm i\pi$, (the ‘phase jump’ at the critical point) can be stabilizing or destabilizing depending on the sign of $U''_c$, since this multiplies the logarithm in (100).

For a neutral wave, with a real dispersion relation, when there is a single critical point, we must have $U''_c = 0$, i.e. the critical point lies at the inflexion point. The eigenfunction is then regular.

If two critical points are present, a neutral wave can be constructed by requiring the stabilizing contribution from one critical point to cancel the destabilizing contribution of the other.
Long-wave inviscid boundary layer stability

• Consider a smooth boundary layer velocity profile $U(y)$ that increases monotonically from $U(0) = 0$ (at the wall) to $U = 1$ in the freestream.

• Let the boundary layer thickness (e.g. displacement thickness, momentum thickness, or distance from the wall to where $U$ is, say 95% of its freestream value) be of $O(1)$.

• Consider disturbances whose wavelengths are long compared with the boundary layer thickness, so let $\alpha = \alpha_0 \epsilon$, $\epsilon \ll 1$.

• Far from the boundary layer, $y \gg 1$, we have $U = 1$, and so in this region the solution takes the form

$$v = Ae^{-\alpha y} = Ae^{-\alpha_0 \epsilon y}$$  \hspace{1cm} (104)

as in (87).
• Therefore, the length scale over which the disturbance decays outside the boundary layer is $O(\epsilon^{-1})$, i.e. of order the disturbance wavelength.

• Behaviour of $v$ inside the boundary layer: at leading order as $\epsilon \to 0$, the Rayleigh equation (38) reduces to

$$ (U - c)v'' - U''v = 0. \quad (105) $$

• By inspection,

$$ v = U - c \quad (106) $$

is a solution of (105).

• A second solution can be found by substituting $v = (U - c)V$ into (105):
• giving

\[ V'' + \frac{2U'}{U - c} V' = 0 \]  \hfill (107)

• (107) can be solved using an integrating factor for \( V' \) leading to a second solution of (105):

\[ v = (U - c) \int \frac{dy}{(U - c)^2}. \]  \hfill (108)

• But this integral has a non-integrable singularity at \( y = y_c \).

• Note that the Laurent series of the integrand about \( y_c \) is

\[ \frac{1}{(U - c)^2} \sim \frac{1}{(U'_c)^2(y - y_c)^2} - \frac{U''_c}{(U'_c)^3(y - y_c)} + O(1). \]  \hfill (109)
We can extract the singular part from the integral:

\[
\int \frac{dy}{(U - c)^2} = \int \left[ \frac{1}{(U - c)^2} - \frac{1}{(U'_c)^2(y - y_c)^2} + \frac{U''_c}{(U'_c)^3(y - y_c)} \right] \\
\quad + \frac{U''}{(U'_c)^2(y - y_c)^2} - \frac{U'_c}{(U'_c)^3(y - y_c)} \ dx \\
\quad = \int \frac{1}{(U - c)^2} - \frac{1}{(U'_c)^2(y - y_c)^2} + \frac{U''_c}{(U'_c)^3(y - y_c)} \ dx \\
\quad - \frac{1}{(U'_c)^2(y - y_c)} - \frac{U'_c}{(U'_c)^3(y - y_c)} \ln(y - y_c) \\
\quad \text{(110)}
\]

where the integrand in (110) is regular.
We choose the second solution to (105) to be

\[ v = (U - c) \left[ \int_0^y \frac{1}{(U - c)^2} - \frac{1}{(U'_c)^2(t - y_c)^2} + \frac{U''_c}{(U'_c)^3(t - y_c)} \, dt - \frac{1}{(U'_c)^2(y - y_c)} - \frac{U''_c}{(U'_c)^3 \ln(y - y_c)} \right] \]

(111)

where the integrand is regular.

- The solution in the main part of the boundary layer is a linear superposition of (106) and (111).
- (106) is the long-wave version of the regular Tollmien solution (96), and (111) has the logarithmic behaviour of the singular Tollmien solution (100).
• In the long-wave limit the piecewise-linear boundary layer had \( c = O(\alpha) \), see (92) (when \( \delta U_1 = O(1) \), as here).

• Does \( c \) follow the same scaling for a smooth profile?

• Try

\[
c = c_0 \epsilon^a
\]  

where \( a \) is some constant to be determined, and \( c_0 = O(1) \).

• If \( a > 0 \), then \( c \) is small, and the critical point is close to the wall because there \( U \) is also small.

• The Taylor series for \( U \) is

\[
U = U'_0 y + U''_0 \frac{y^2}{2} + \ldots
\]  

where \( U'_0 = U'(0) \), etc.
• Setting $c = U(y_c)$, using (112) and the leading term of (113), gives

\[ y_c \sim \frac{c_0 \epsilon^a}{U_0'} . \]

(114)

• The regular part of the solution, (106), varies over the same scale as $U$, i.e. on the scale $y = O(1)$.

• The singular part of the solution, (111), varies most sharply when $y$ is comparable to $y_c$, i.e. on the scale $y = O(\epsilon^a)$.

• And outside the boundary layer, the disturbance decays on the scale $y = O(\epsilon^{-1})$, see (104).
Inviscid triple-decked disturbances

In summary, the disturbance varies on three different length scales in different parts of the flow:

- \[ y = O(\epsilon^{-1}), \text{ upper layer, exponential behaviour} \]
- \[ y = O(1), \text{ main layer} \]
- \[ y = O(\epsilon^a), \text{ critical layer, logarithmic behaviour} \]
Matched asymptotic expansions

- The method of matched asymptotic expansions involves obtaining solutions in each layer, then matching them to produce a solution uniformly valid across all layers (we shall use van Dyke’s matching rule).

- We shall work directly with the inviscid version of the linearized continuity and momentum equations (28) – (30), rather than the Rayleigh equation:

\[
\begin{align*}
    i\alpha u + v' &= 0 \quad (115) \\
    -i\omega u + i\alpha Uu + U'v &= -i\alpha p \quad (116) \\
    -i\omega v + i\alpha Uv &= -p'. \quad (117)
\end{align*}
\]

- These equations take different, simplified forms in each layer.

- The thickness of the critical layer, \(O(\epsilon^a)\), is determined as part of the solution process.
• The idea of dominant balance is central to the method.
• The order of magnitudes (in terms of powers of $\epsilon$) of $u$, $v$ and $p$ can be different in each layer.
• These various orders of magnitude, and the critical layer thickness, are all determined in the leading order calculation.
• We shall start with the leading order calculation, then add higher order terms later.
• Note that
  \[ \omega = c\alpha = c_0\alpha_0\epsilon^{a+1} \tag{118} \]
  using $\alpha = \alpha_0\epsilon$ and (112).
Upper layer

- Define an upper-layer variable $y = y_u / \epsilon$, where $y_u = O(1)$ in the upper layer.
- In the upper layer $U = 1 \Rightarrow U' = 0$.
- Let
  $u(y) = u_{u0}(y_u), \quad v(y) = v_{u0}(y_u), \quad p(y) = p_{u0}(y_u)$. (119)
  be the upper layer variables to be solved for.
- Note, e.g.,
  $$v' = \frac{dv}{dy} = \frac{dv_{u0}}{dy} = \frac{dv_{u0}}{dy_u} \frac{dy_u}{dy} = \epsilon v'_{u0}. \quad (120)$$
- Substitute (118), (119), $U = 1$ and $\alpha = \alpha_0 \epsilon$ into (115) – (117). At $O(\epsilon)$:
  $i\alpha_0 u_{u0} + v'_{u0} = 0 \quad (121)$
  $i\alpha_0 u_{u0} + i\alpha_0 p_{u0} = 0 \quad (122)$
  $i\alpha_0 v_{u0} + p'_{u0} = 0. \quad (123)$
• Eliminating $u_0$ and $v_0$ from (121) – (123) gives

$$p''_u - \alpha_0^2 p_u = 0.$$  \hspace{1cm} (124)

• The solution to (124) that decays far from the wall, and hence $v_u$ from (123), is

$$p_{u0} = P_{u0} e^{-\alpha_0 y_u}$$  \hspace{1cm} (125)

$$v_{u0} = -i P_{u0} e^{-\alpha_0 y_u}$$  \hspace{1cm} (126)

where $P_{u0}$ is a constant.

• These upper-layer solutions will be matched to the main-layer solutions.
Main layer

- The main-layer variables are

\[ u = \epsilon^{-1} u_m(y), \quad v = v_m(y), \quad p = p_m(y). \]  \hfill (127)

- The magnitudes of \( v \) and \( p \) in the main layer are the same as in the upper layer so they can be matched with the upper layer.

- The principle of dominant balance requires \( u = O(\epsilon^{-1}) \), so that \( u \) appears at leading order in the continuity equation:

- Substituting (118), (127) and \( \alpha = \alpha_0 \epsilon \) into (115) – (117) gives at leading order

\[ i\alpha_0 u_m + v'_m = 0 \]  \hfill (128)
\[ i\alpha_0 U u_m + U' v_m = 0 \]  \hfill (129)
\[ p'_m = 0. \]  \hfill (130)
Therefore, in the main layer the velocity and pressure fields decouple.

The solutions of (128) – (130) are

\[ p_{m0} = P_{m0} \quad (131) \]
\[ v_{m0} = A_{m0} U \quad (132) \]

where \( A_{m0} \) and \( P_{m0} \) are constants.

These main-layer solutions are to be matched with both the upper layer and the critical layer.

Note that \( v_{m0} \sim A_{m0} U_0' y \) for small \( y \), and hence \( v_{m0} = O(\epsilon^a) \) in the critical layer.
Critical layer

- Define a critical layer variable $y = \epsilon^a Y$, where $Y = O(1)$ in the critical layer.
- In the critical layer, the basic flow takes the form
  \[
  U = \epsilon^a U_0' Y + \epsilon^{2a} U_0'' \frac{Y^2}{2} + \ldots, \quad U' = U_0' + \epsilon^a U_0'' Y + \ldots \quad (133)
  \]
- Let
  \[
  u = \epsilon^{-1} u_{c0}(Y), \quad v = \epsilon^a v_{c0}(Y), \quad p = p_{c0}(Y). \quad (134)
  \]
- The magnitudes of $v$ and $p$ are chosen so they can be matched to the main-layer solutions.
- **Dominant balance** requires $u = O(\epsilon^{-1})$ so that $u_{c0}$ appears at leading order in the continuity equation.
• Substituting (118), (133), (134) and $\alpha = \alpha_0 \epsilon$ into (115) – (117), and neglecting the unambiguously smaller terms, gives

$$i\alpha_0 u_{c0} + \nu'_{c0} = 0 \quad (135)$$

$$-ic_0\alpha_0 \epsilon^a u_{c0} + i\alpha_0 \epsilon^a U'_0 Y u_{c0} + \epsilon^a U'_0 \nu_{c0} + i\alpha_0 \epsilon p_{c0} = 0 \quad (136)$$

$$p'_{c0} = 0. \quad (137)$$

• The principle of dominant balance gives $a = 1$.
• This choice of $a$ couples the velocity and pressure fields at leading order.
• $a = 1 \Rightarrow c = O(\epsilon) \Rightarrow c = O(\alpha)$, i.e. the same scaling as for the piecewise-linear boundary layer, (92).
• With \( a = 1 \), (135) – (137) become

\[
\begin{align*}
\text{i}\alpha_0 u_c^0 + v'_c^0 &= 0 \quad (138) \\
-\text{i}c_0\alpha_0 u_c^0 + \text{i}\alpha_0 U'_0 Y u_c^0 + U'_0 v_c^0 + \text{i}\alpha_0 p_c^0 &= 0 \quad (139) \\
p'_c^0 &= 0. \quad (140)
\end{align*}
\]

• The general solution to (138) – (140) is

\[
\begin{align*}
p_c^0 &= P_c^0 \quad (141) \\
v_c^0 &= B_c^0 (U'_0 Y - c_0) - \frac{\text{i}\alpha_0 P_c^0}{c_0} Y \quad (142)
\end{align*}
\]

where \( P_c^0 \) and \( B_c^0 \) are constants.
Matching between layers

- We have found the thickness of the critical layer and the solutions in all three layers.
- Now match between the layers to find relations between the constants $P_{u0}$, $A_{m0}$, $P_{m0}$, $B_{c0}$ and $P_{c0}$.
- Essentially, this requires the solutions in adjacent layers to merge into one-another in an over-lap region between the two layers.
- We shall use van Dyke’s (1964) matching rule.
‘Imperceptible blending’ – M.C. Escher
• In van Dyke’s matching rule, the solution in each layer is written in terms of the variable in the next layer, e.g. 
\( v_{u0}(y_u) = v_{u0}(\epsilon y) \), and then expanded as a series as \( \epsilon \rightarrow 0 \).
• We write the matching of \( v \) between the upper and main layers as

\[
H_0\{v_{u0}(\epsilon y)\} = H_0\{v_{m0}(y_u/\epsilon)\}
\]

(143)

where \( H_n\{\cdot\} \) means expand its argument in powers of \( \epsilon \) up to and including \( O(\epsilon^n) \).
• Substituting (126) and (132) into (143) we have

\[
H_0\{v_{u0}(\epsilon y)\} = H_0\{-iP_{u0}e^{-\alpha_0\epsilon y}\} \\
= H_0\{-iP_{u0}(1 - \epsilon\alpha_0 y + \ldots)\} = -iP_{u0} \quad (144)
\]

\[
H_0\{v_{m0}(y_u/\epsilon)\} = H_0\{A_{m0}U(y_u/\epsilon)\} = H_0\{A_{m0}\} = A_{m0} \quad (145)
\]

\[
\Rightarrow -iP_{u0} = A_{m0}. \quad (146)
\]
• To match pressures between upper and main layers, substitute (125) and (131) into

\[ H_0\{p_{u0}(\epsilon y)\} = H_0\{p_{m0}(y_u/\epsilon)\} \]  \hspace{1cm} (147)

\[ \Rightarrow P_{u0} = P_{m0}. \]  \hspace{1cm} (148)

• To match pressures between main and critical layers, substitute (131) and (141) into

\[ H_0\{p_{m0}(\epsilon Y)\} = H_0\{p_{c0}(y/\epsilon)\} \]  \hspace{1cm} (149)

\[ \Rightarrow P_{m0} = P_{c0}. \]  \hspace{1cm} (150)
To match velocities between main and critical layers, substitute (132) and (142) into

\[ H_1\{v_{m0}(\epsilon Y)\} = H_0\{\epsilon v_{c0}(y/\epsilon)\} \]  \hspace{1cm} (151)

\[ \Rightarrow H_1\{A_{m0} U(\epsilon Y)\} = H_0 \left\{ \epsilon \left[ B_{c0} \left( U_0' \frac{y}{\epsilon} - c_0 \right) - \frac{i\alpha_0 P_{c0} y}{c_0} \right] \right\} \]

\[ \Rightarrow A_{m0} U_0' \epsilon Y = B_{c0} U_0' y - \frac{i\alpha_0 P_{c0} y}{c_0} \]

\[ \Rightarrow A_{m0} U_0' = B_{c0} U_0' - \frac{i\alpha_0 P_{c0}}{c_0}. \]  \hspace{1cm} (152)

Eliminating the constants \( P_{u0}, A_{m0} \) and \( P_{m0} \) from (146), (148), (150) and (152) gives

\[ B_{c0} = iP_{c0} \left( \frac{\alpha_0}{U'_0 c_0} - 1 \right). \]  \hspace{1cm} (153)

Substituting (153) into (142) gives

\[ v_{c0} = -iP_{c0} \left( U'_0 Y - c_0 + \frac{\alpha_0}{U'_0} \right). \]  \hspace{1cm} (154)
• (154) can be interpreted as a linear combination of the first terms from Tollmien’s solutions (96) and (100).

• Van Dyke’s matching rule gives the particular linear combination that decays exponentially as $y \to \infty$.

• Applying the wall boundary-condition $v_{c0}(0) = 0$ to (154) gives the leading-order inviscid dispersion relation:

$$c_0 = \frac{\alpha_0}{U'_0} \Rightarrow c = \frac{\alpha}{U'_0}.$$  \hspace{1cm} (155)

• The piecewise-linear boundary layer (85) has a velocity gradient in the lower segment of $U_1/\delta = U'_1 \Rightarrow \delta = U_1/U'_1$, hence the long-wave result (92) can be written $c \sim \alpha U^2_1/U'_1$.

• (155) was derived for a boundary layer with $U = 1$ in the freestream, corresponding to $U_1 = 1$, and the piecewise-linear long-wave result is then $c \sim \alpha/U'_1$.

• Therefore, for long waves, piecewise-linear theory agrees best with a smooth $U(y)$ if velocity gradients at the wall are equal.
Second-order long-wave theory

- Having obtained the leading-order long-wave solution, and established the matched asymptotic framework for the problem, it is straightforward, in principle, to extend the analysis to higher order.
- We shall carry out the second-order calculation to capture the phase jump at the critical layer.
- This phase jump controls the stability/instability — it is a qualitatively new effect, and so is worth the effort...
- Making use of the result $a = 1$, and in anticipation of the logarithmic behaviour of (111), we try the expansion
  \[
  \omega = c_0 \alpha_0 \epsilon^2 + c_{1L} \alpha_0 \epsilon^3 \ln \epsilon + c_1 \alpha_0 \epsilon^3 + \ldots
  \]
  (156)
- We follow the same steps as before, but there are more terms to manipulate.
Upper layer

• The upper-layer variables (119) are now expanded as

\begin{align*}
    u &= u_0 + u_1 L \epsilon \ln \epsilon + u_1 \epsilon + \ldots \\
    \nu &= \nu_0 + \nu_1 L \epsilon \ln \epsilon + \nu_1 \epsilon + \ldots \\
    p &= p_0 + p_1 L \epsilon \ln \epsilon + p_1 \epsilon + \ldots 
\end{align*} \quad (157) \quad (158) \quad (159)

where all the new variables are functions of \( y_u \).

• Sub. \( \alpha = \alpha_0 \epsilon \), (156) – (159), \( U = 1 \), into (115) – (117).

• At leading order (121) – (123) are reproduced.

• At next order we find

\begin{align*}
    i \alpha_0 u_1 L + \nu'_1 L &= 0 \quad (160) \\
    i \alpha_0 u_1 L + i \alpha_0 p_1 L &= 0 \quad (161) \\
    i \alpha_0 \nu_1 L + p'_1 L &= 0. \quad (162)
\end{align*}
• Equations (160) – (162) are essentially the same as (121) – (123), so their solutions can be written down immediately:

\[ p_{u1L} = P_{u1L}e^{-\alpha_0 y_u} \]  \hspace{1cm} (163)
\[ v_{u1L} = -iP_{u1L}e^{-\alpha_0 y_u} \]  \hspace{1cm} (164)

where \( P_{u1L} \) is a constant.

• At the next order in the expansion we find forced equations

\[ i\alpha_0 u_{u1} + v'_{u1} = 0 \]  \hspace{1cm} (165)
\[ i\alpha_0 u_{u1} + i\alpha_0 p_{u1} = ic_0\alpha_0 u_{u0} \]  \hspace{1cm} (166)
\[ i\alpha_0 v_{u1} + p'_{u1} = ic_0\alpha_0 v_{u0}. \]  \hspace{1cm} (167)
• Eliminating $u_{u1}$ and $v_{u1}$ from (165) – (167), making use of (121) – (123) as necessary, gives

$$p''_{u1} - \alpha_0^2 p_{u1} = 0. \quad (168)$$

• The solution to (168) that decays far from the wall, and the solution for $v_{u1}$, from (126) and (167), are

$$p_{u1} = P_{u1} e^{-\alpha_0 y_u} \quad (169)$$

$$v_{u1} = -i (P_{u1} + c_0 P_{u0}) e^{-\alpha_0 y_u} \quad (170)$$

where $P_{u1}$ is a constant.
The main-layer variables (127) are now expanded as

\[ u = u_m^0 \epsilon^{-1} + u_m^1 L \ln \epsilon + u_m^1 + \ldots \]  
\[ v = v_m^0 + v_m^1 L \epsilon \ln \epsilon + v_m^1 \epsilon + \ldots \]  
\[ p = p_m^0 + p_m^1 L \epsilon \ln \epsilon + p_m^1 \epsilon + \ldots \]  

where all the new variables are functions of \( y \).

- Substitute \( \alpha = \alpha_0 \epsilon \), (156), (171) – (173) into (115) – (117).
- At leading order (128) – (130) are reproduced.
- At the next order, the equations for \( u_m^1 L \), \( v_m^1 L \) and \( p_m^1 L \) have the same form as (128) – (130).
Therefore, their solutions have the same form as (131) and (132):

\[
p_{m1L} = P_{m1L} \quad \text{(174)}
\]
\[
\nu_{m1L} = A_{m1L} U \quad \text{(175)}
\]

where \( A_{m1L} \) and \( P_{m1L} \) are constants.

At the next order, forced equations appear:

\[
i\alpha_0 u_{m1} + \nu_{m1}' = 0 \quad \text{(176)}
\]
\[
i\alpha_0 U u_{m1} + U' \nu_{m1} = i c_0 \alpha_0 u_{m0} - i \alpha_0 p_{m0} \quad \text{(177)}
\]
\[
p_{m1}' = -i \alpha_0 U \nu_{m0} \quad \text{(178)}
\]
Substituting the leading order solutions (131) and (132) into (176) – (178), and eliminating $u_{m1}$ gives

$$v'_{m1} - \frac{U'}{U}v_{m1} = c_0 A_{m0} \frac{U'}{U} + \frac{i\alpha_0 P_{m0}}{U}$$  \hspace{1cm} (179)$$

$$p'_{m1} = -i\alpha_0 A_{m0} U^2.$$  \hspace{1cm} (180)

A solution to (180) can be written

$$p_{m1} = -i\alpha_0 A_{m0} \int U^2 \, dy.$$  \hspace{1cm} (181)

However, the integral in (181) is divergent as $y \rightarrow \infty$, and the behaviour of $p_{m1}$ in this limit will be needed when matching to the upper-layer solution.
• Therefore, we remove the divergent part of the integral, and choose instead

\[ p_{m1} = P_{m1} - i\alpha_0 A_0 \left( \int_0^y U^2 - 1 \, dt + y \right). \]  

(182)

• A solution to (179) can be obtained using an integrating factor:

\[ v_{m1} = A_{m1} U - c_0 A_0 + i\alpha_0 P_0 U \int \frac{dy}{U^2}. \]  

(183)

• However, the integral in (183) is divergent both as \( y \to 0 \) and \( y \to \infty \), and we will need the behaviour of \( v_{m1} \) in both these limits when matching with the critical-layer solutions and upper-layer solutions.
Therefore, we remove the divergent parts of the integral, and choose instead

\[ v_{m1} = A_{m1} U - c_0 A_{m0} + i \alpha_0 P_{m0} U \left[ \int_0^y \frac{1}{U^2} - \frac{1}{(U'_0)^2 t^2} \right. \\
\left. + \frac{U''_0}{(U'_0)^3 t} - 1 \, dt - \frac{1}{(U'_0)^2 y} - \frac{U''_0}{(U'_0)^3} \ln(y) + y \right] \]

(184)

(similar to the treatment of the integral in (111)).

This solution captures the logarithmic behaviour of the singular Tollmien solution (100).
The critical-layer variables (134) are now expanded as

\[ u = u_{c0} \epsilon^{-1} + u_{c1L} \ln \epsilon + u_{c1} + \ldots \quad (185) \]
\[ v = v_{c0} \epsilon + v_{c1L} \epsilon^2 \ln \epsilon + v_{c1} \epsilon^2 + \ldots \quad (186) \]
\[ p = p_{c0} + p_{c1L} \epsilon \ln \epsilon + p_{c1} \epsilon + \ldots \quad (187) \]

where all the new variables are functions of \( Y \), where \( y = \epsilon Y \).

The basic flow in the critical layer is

\[ U = \epsilon U'_{0} Y + \epsilon^2 U''_{0} \frac{Y^2}{2} + \ldots, \quad U' = U'_0 + \epsilon U''_{0} Y + \ldots \quad (188) \]

Substitute \( \alpha = \alpha_0 \epsilon \), (156) and (185) – (188) into (115) – (117).

At leading order (138) – (140) are reproduced.
• At the order where $\ln \epsilon$ appears, the equations are

\begin{align}
  i\alpha_0 u_{c1L} + \nu'_{c1L} &= 0 \quad (189) \\
  -ic_0\alpha_0 u_{c1L} + i\alpha_0 U'Y u_{c1L} + U'_{c1L} + i\alpha_0 p_{c1L} &= i\alpha_0 c_{1L} u_{c0} \quad (190) \\
  p'_{c1L} &= 0. \quad (191)
\end{align}

• Substituting the leading order solution (154) into (189) – (191) gives solutions

\begin{align}
  p_{c1L} &= P_{c1L} \quad (192) \\
  \nu_{c1L} &= B_{c1L} (U'_{0} Y - c_{0}) - \frac{i(\alpha_0 P_{c1L} - U'_{0} c_{1L} P_{c0})}{c_{0}} Y. \quad (193)
\end{align}

where $P_{c1L}$ and $B_{c1L}$ are constants.
At the next order basic flow curvature terms appear:

\[ i\alpha_0 u_{c1} + v'_{c1} = 0 \quad (194) \]

\[-ic_0\alpha_0 u_{c1} + i\alpha_0 U'_0 Y u_{c1} + U'_0 v_{c1} + i\alpha_0 p_{c1} = ic_1\alpha_0 u_{c0} - \]

\[ \frac{i\alpha_0 U''_0 Y^2}{2} u_{c0} - U''_0 Y v_{c0} \quad (195) \]

\[ p'_{c1} = 0. \quad (196) \]
Substituting the leading-order result (154) into (195), we obtain the general solution

\[ p_{c_1} = P_{c_1} \]  \hspace{1cm} (197)

\[ v_{c_1} = B_{c_1} \left( U'_0 Y - c_0 \right) - \frac{i(\alpha_0 P_{c_1} - U'_0 c_1 P_{c_0})}{c_0} Y \]

\[ - \frac{iU''_{0_0}}{(U'_0)^2} \left[ \frac{(U'_0 Y)^2}{2} - \alpha_0 Y \right] \]

\[ + \alpha_0 \left( Y - \frac{c_0}{U'_0} \right) \ln \left( Y - \frac{c_0}{U'_0} \right) \]  \hspace{1cm} (198)

where \( P_{c_1} \) and \( B_{c_1} \) are constants.
Matching upper and main layers

- From (159) and (173), van Dyke’s matching rule for the pressure for these layers is

\[ H_1 \left\{ p_{u0}(\epsilon y) + p_{u1L}(\epsilon y)\epsilon \ln \epsilon + p_{u1}(\epsilon y)\epsilon \right\} = H_1 \left\{ p_{m0}(y_u/\epsilon) + p_{m1L}(y_u/\epsilon)\epsilon \ln \epsilon + p_{m1}(y_u/\epsilon)\epsilon \right\}. \]  

(199)

- Substitute (125), (163), (169), (131), (174) and (182) into (199) and equate coefficients of powers of \( \epsilon \).

- At leading order we reproduce the leading order matching result (148), and at next orders find

\[ P_{u1L} = P_{m1L} \]  

(200)

\[ P_{u1} = P_{m1} - \alpha_0 I_1 P_{c0} \]  

(201)

where \( I_1 = \int_0^\infty U^2 - 1 \, dy \), and leading order constants have been expressed in terms of \( P_{c0} \) via the leading order matching results (146), (148), (150) and (152).
• From (158) and (172), van Dyke’s matching rule for the velocity for these layers is

\[ H_1 \left\{ v_{u0}(\epsilon y) + v_{u1L}(\epsilon y)\epsilon \ln \epsilon + v_{u1}(\epsilon y)\epsilon \right\} = 
H_1 \left\{ v_{m0}(y_u/\epsilon) + v_{m1L}(y_u/\epsilon)\epsilon \ln \epsilon + v_{m1}(y_u/\epsilon)\epsilon \right\}. \]  

(202)

• Substitute (126), (164), (170), (132), (175) and (184) into (202) and equate coefficients of powers of \( \epsilon \).

• At leading order we reproduce the leading order matching result (146), and at next orders find

\[-iP_{u1L} = A_{m1L} \]  

(203)

\[-iP_{u1} = A_{m1} + iP_{c0}(2c_0 + \alpha_0 l_2) \]  

(204)

where

\[ l_2 = \lim_{y \to \infty} \left[ \int_0^y \frac{1}{U^2} - \frac{1}{(U')^2 t^2} + \frac{U''}{(U')^3 t} - 1 \, dt - \frac{U''}{(U')^3} \ln(y) \right]. \]  

(205)
Matching main and critical layers

- From (173) and (187), van Dyke’s matching rule for the pressure for these layers is

\[ H_1 \left\{ p_{m0}(\epsilon Y) + p_{m1L}(\epsilon Y)\epsilon \ln \epsilon + p_{m1}(\epsilon Y)\epsilon \right\} = H_1 \left\{ p_{c0}(y/\epsilon) + p_{c1L}(y/\epsilon)\epsilon \ln \epsilon + p_{c1}(y/\epsilon)\epsilon \right\}. \] (206)

- Substitute (131), (174), (182), (141), (192) and (197) into (206) and equate coefficients of powers of \( \epsilon \).
- At leading order we reproduce the leading order matching result (150), and at next orders find

\[ P_{m1L} = P_{c1L} \] (207)
\[ P_{m1} = P_{c1}. \] (208)
From (172) and (186), van Dyke’s matching rule for the velocity for these layers is

\[
H_2 \left\{ v_{m0}(\epsilon Y) + v_{m1L}(\epsilon Y)\epsilon \ln \epsilon + v_{m1}(\epsilon Y)\epsilon \right\} =
H_1 \left\{ v_{c0}(y/\epsilon)\epsilon + v_{c1L}(y/\epsilon)\epsilon^2 \ln \epsilon + v_{c1}(y/\epsilon)\epsilon^2 \right\}.
\]

Note from (184) that \( v_{m1} \) has a term of order \( y \ln y \) for small \( y \), and from (198) that \( v_{c1} \) has a term of order \( Y \ln Y \) for large \( Y \).

Converting between \( y \) and \( Y \) generates a \( \ln \epsilon \) term:

\[
y \ln y = \epsilon Y \ln(\epsilon Y) = \epsilon Y \ln \epsilon + \epsilon Y \ln Y.
\]

The existence of this \( \ln \epsilon \) term forces the presence of the \( \ln \epsilon \) terms in the expansions (156), (157) – (159), (171) – (173) and (185) – (187).
• Substitute (132), (154), (175), (184), (193) and (198) into (209) and equate coefficients of powers of $\epsilon$.

• At leading order we reproduce the leading order matching result (152), and at next orders find

$$A_{m1L} U'_0 = B_{c1L} U'_0 - \frac{i(\alpha_0 P_{c1L} - c_{1L} U'_0 P_{c0})}{c_0} + \frac{i\alpha_0 U''_0 P_{c0}}{(U'_0)^2}$$

(211)

$$A_{m1} U'_0 = B_{c1} U'_0 - \frac{i(\alpha_0 P_{c1} - c_1 U'_0 P_{c0})}{c_0} + \frac{3i\alpha_0 U''_0 P_{c0}}{2(U'_0)^2}.$$

(212)
Dispersion relations

- Eliminating $P_{u1L}$, $P_{m1L}$, $A_{m1L}$ and $P_{c1L}$ from (200), (203), (207) and (211), and using (155) for $c_0$, gives

$$B_{c1L} = -rac{iU_0'}{\alpha_0^2} P_{c0} \left( c_{1L} + \frac{\alpha_0^2 U_0''}{(U_0')^4} \right).$$  \hspace{1cm} (213)

- Substituting (213) into (193) gives

$$v_{c1L} = -iU_0' Y \left( P_{c1L} + \frac{\alpha_0 U_0'' P_{c0}}{(U_0')^3} \right) + iP_{c0} \left( c_{1L} + \frac{\alpha_0^2 U_0''}{(U_0')^4} \right).$$ \hspace{1cm} (214)

- Applying the wall boundary condition $v_{c1L}(0) = 0$ then gives the inviscid dispersion relation

$$c_{1L} = -\frac{\alpha_0^2 U_0''}{(U_0')^4}.$$  \hspace{1cm} (215)
Eliminating $P_{u1}$, $P_{m1}$, $A_{m1}$ and $P_{c1}$ from (201), (204), (208) and (212), and using (155) for $c_0$, gives

$$B_{c1} = -i\alpha_0 P_{c0} \left( \frac{U'_0}{\alpha_0^2} c_1 - l_1 + l_2 + \frac{2}{U'_0} + \frac{3U''_0}{2(U'_0)^3} \right).$$

Substituting (216) into (198) gives

$$v_{c1} = i\alpha_0^2 P_{c0} \left( \frac{c_1}{\alpha_0^2} - \frac{l_1 - l_2}{U'_0} + \frac{2}{(U'_0)^2} + \frac{3U''_0}{2(U'_0)^4} \right)$$

$$-iU'_0 Y \left[ P_{c1} - \left( l_1 - l_2 - \frac{2}{U'_0} - \frac{3U''_0}{2(U'_0)^3} \right) \alpha_0 P_{c0} \right] - \frac{iU''_0 P_{c0}}{(U'_0)^2} \left[ \frac{(U'_0 Y)^2}{2} - \alpha_0 Y + \alpha_0 \left( Y - \frac{\alpha_0}{(U'_0)^2} \right) \ln \left( Y - \frac{\alpha_0}{(U'_0)^2} \right) \right].$$

(217)
Applying the wall boundary condition $v_{c1}(0) = 0$ to (217), and using (102), then gives the inviscid dispersion relation

$$\alpha_0^2 \left\{ \frac{l_1 - l_2}{U'_0} - \frac{2}{(U'_0)^2} - \frac{U''_0}{(U'_0)^4} \left[ \frac{3}{2} + \ln \left( \frac{\alpha_0}{(U'_0)^2} \right) - i\pi \right] \right\}. \quad (218)$$

Prandtl’s boundary layer equations show that $U''_0 = \partial P/\partial x$, where $P$ is the basic flow pressure in the freestream (see section 3).
• Therefore,

\[
\text{Accelerating freestream} \implies \frac{\partial P}{\partial x} < 0 \implies U_0'' < 0 \\
\implies \text{stable.}
\]

\[
\text{Decelerating freestream} \implies \frac{\partial P}{\partial x} > 0 \implies U_0'' > 0 \\
\implies \text{unstable.}
\]

• Note \( U'' < 0 \) for large \( y \).

• Accelerating boundary layers have no inflexion points and are stable in agreement with Rayleigh’s inflexion point theorem.

• Decelerating boundary layers have an inflexion point and are unstable.

• Over a typical wing, the flow accelerates near the leading edge, and decelerates further downstream.

• Laminar flow can exist near the leading edge, but transition to turbulence occurs further downstream.
The Blasius boundary layer

• The case of constant freestream velocity, corresponding to flow over a flat plate aligned with the freestream (Blasius flow), has $U''_0 = 0$, and so is neutral at this order in the calculation.

• It can be shown that near the wall the $y$-dependence of Blasius flow takes the form

$$U = U'_0 y - \frac{(U'_0)^2}{48} y^4 + \ldots$$  \hspace{1cm} (219)

(using suitable non-dimensional variables, see section 3).

• The singular Tollmien solution (100) shows that the coefficient of the phase jump $i\pi$ produced by the logarithm is always proportional to $U''_c$.

• (219) shows that $U'' = O(y^2) = O(\epsilon^2)$ in the critical layer for Blasius flow.
Therefore, (156) is replaced by

\[ \omega = c_0 \alpha_0 \epsilon^2 + c_1 \alpha_0 \epsilon^3 + c_2 \alpha_0 \epsilon^4 + c_3L \alpha_0 \epsilon^5 \ln \epsilon + c_3 \alpha_0 \epsilon^5 + \ldots \]  

(220)

Our interest lies in the stability characteristics, which are due to the critical-layer phase-jump \( i\pi \) produced by the logarithm, which will appear in \( c_3 \).

After much algebra, it can be shown that

\[ c_3 = \ldots - \frac{i\pi \alpha_0^4}{4(U_0')^6}. \]  

(221)

Therefore, the Blasius boundary layer is stable.

In agreement with Rayleigh’s theorem, since the inflexion point is at the wall, \( U_0'' = 0 \).

But the decay rate is very small for long waves: \( \text{Im}(\omega) = O(\epsilon^5) \).